SECTION B

THEORETICAL AND NUMERICAL ANALYSIS OF TWO REGIME FLOW TOWARD WELLS

by

P. S. Huyakorn

Summary

It has long been recognised that the flow in the immediate vicinity of a well is associated with steep hydraulic gradients, which may violate Darcy's law. Experimental investigations have shown that departure from the linear relationship starts at a range of Reynolds number between 1 and 10, depending upon the distribution of grain size, particle shape, and the degree of compaction of the aquifer material. (Todd, 1959).

Relatively few analyses have been made of the flow towards wells in which the non-Darcy flow in the zone at close proximity to the well boundary has been taken into account. It is only recently that the significance of this zone has been emphasised. In many cases, computed flow rates based on Darcy's law are significantly greater than that actually measured for fixed values of the well drawdown. The computation of drawdowns near the well may also be grossly in error.

Two difficulties that confronted earlier workers were:-

(i) the two regime nature of the flow;

(ii) the non-linearity of the field equation, which prohibits solutions by analytical techniques.

Furthermore the transition from one flow regime to the second regime has not been clearly defined.

In this section of the report, the author has attempted to apply the method of continuum mechanics to formulate the generalised problem of three-dimensional well flow, and to apply the finite element technique to solve the governing field equations for various flow cases that frequently occur in practice. Methods of evaluating aquifer properties taking account of the non-Darcy effects are being developed and means for predicting and allowing for these effects are being studied. Results of these studies will be reported at a later date.

Table of Contents

Table of Contents (cont'd.)

List of Figures

1. Introduction

1.1 Literature Review

1.1.1 Well Hydraulics

The hydraulics of well flow based on Darcy's law have been extensively studied. The simple linear relationship between the hydraulic gradient and the macroscopic flow velocity gives rise to the linear governing equation, which has been solved by analytical methods for a number of flow systems where the aquifer is uniform and the boundary conditions are relatively simple. A great deal of information on the solutions and methods of evaluating the aquifer properties has been published. An annotated bibliography has been presented by Huyakorn and Dudgeon (1972).

It has long been recognised that the flow in the immediate vicinity of the well is associated with steep hydraulic gradients, which may violate Darcy's law. The non-linear behaviour prevailing in this zone may have a considerable influence on both the discharge and specific capacity of the well. Field observations often reveal that the computation of drawdown in close proximity to well boundary is grossly in error.

The significance of the non-linear effect was first pointed out by Jacob (1947) who proposed a hypothetical well-aquifer model incorporating turbulent flow. He postulated that the flow through the aquifer follows the linear velocity-gradient relation right up to a boundary surface generated by what he termed the "effective well radius". This radius is defined as the distance from the axis of the well at which the computed drawdown based on Darcy's law equals the actual drawdown just outside the screen. Inside the "effective well radius" the flow was assumed to be fully turbulent, and the headloss was considered to be made up of losses due to turbulence in the formation as well as into and inside the well.

To account for the effects of these losses on the discharge, he suggested the following equation:

$$s_w = BQ + CQ^2 \qquad (1\text{-}1)$$

where s_w is the well drawdown, Q is the discharge, B and C are empirical constants.

Rorabaugh (1953) proposed a similar model but a slightly different equation in which he expressed the drawdown in the well as

$$s_w = BQ + CQ^n \qquad (1-2)$$

where n is the exponential constant having a value between 1 and 2.

Whilst it has been found that the above equations fit many observed data, certain doubts still exist regarding their general applicability. The empirical constants, B and C, have to be determined by the step draw-down test. Neither equation describes the drawdown distribution inside the effective well radius nor do they determine under what conditions turbulence and hence non-Darcy flow is likely to exist and be of practical significance.

To determine whether or not non-Darcy flow is present or may be expected to occur under a given condition, Mogg (1959) proposed the Reynolds number as a criterion. The Reynolds number, $\mathbb{R}$, was defined as

$$\mathbb{R} = \frac{Vd_{70}}{\nu} \qquad (1-3)$$

where V is the macroscopic bulk velocity, ν is the kinematic viscosity of groundwater at the prevailing temperature, and d_{70} is the characteristic grain diameter as obtained by sieve analysis such that 70% by weight of the sample is of coarser size.

From the results of permeability tests on unconsolidated samples of sands and gravels, Mogg found that the linear velocity-gradient relationship holds provided $\mathbb{R}$ is less than 10. At higher values of $\mathbb{R}$ the head loss varies with the velocity raised to a power between 1 and 2. Using these test data, the curve of the cone of depression was computed for steady radial flow toward a well fully penetrating a uniform aquifer material of similar characteristics to the samples tested. A semi-log plot showed that the curve was non-linear in the turbulent zone near the well boundary and the non-linearity became more pronounced as the boundary was approached. Mogg also concluded that an important factor appearing to reduce the effect of turbulence is the maximum development of the formation material adjacent to the well.

The study by Mogg merely provides a good starting point for later workers. More accurate accounts of the non-linear effects are still necessary for a proper choice of the well dimensions, screens and gravel pack materials.

Engelund (1953) was the first to carry out a more general theoretical investigation into two-dimensional flow. He employed the following equation to describe both Darcy and non-Darcy flow through uniform porous media.

$$\vec{\nabla} h = - F(|V|)\ \vec{V} \tag{1-4}$$

where $\vec{\nabla} h$ is the hydraulic gradient vector, $\vec{V}$ is the velocity vector, and $F(|V|)$ is a scalar function of the absolute velocity $|V|$ and the medium properties.

The function $F(|V|)$ is expressed by

$$F(|V|) = 1/K; \text{for } \mathbb{R} \leqslant \mathbb{R}_{cr} \tag{1-5a}$$

$$F(|V|) = a + b\,|V|; \text{for } \mathbb{R} > \mathbb{R}_{cr} \tag{1-5b}$$

where K is the coefficient of hydraulic conductivity of the medium, a and b are termed "linear and non-linear coefficients of hydraulic resistance", and $\mathbb{R}_{cr}$ is termed the "critical Reynolds number".

By combining equation (1-4) with the continuity equation for two-dimensional steady flow, Engelund obtained a generalised governing equation which is valid for both flow regimes, namely Darcy and non-Darcy. He transformed this equation into a linear form by employing the technique of conformal transformation, derived an equivalent variational form of the equation, and established the concept of the rate of dissipation of hydraulic energy for porous media flow.

Engelund was able to solve the linearised equation analytically for a few limiting cases of flow at high Reynolds number. The major difficulties he encountered were:-

(i) the two regime nature of the flow;

(ii) the complex form of the governing equation in the non-Darcy zone, which prohibits mathematical solutions unless the boundary conditions are made ideally simple.

To avoid these difficulties, the method of finite elements is employed in this study. Using this procedure, complex boundary conditions and non-uniform aquifers can be treated.

1.1.2 Application of the Finite Element Method

The application of the finite element technique to complex groundwater flow has become increasingly popular in recent years.

The development of generalised variational principles on which the technique is usually based, the improvement of numerical methods for solving large systems of algebraic equations, and the progressively greater capacity of modern digital computers all help to make possible solutions of many previously intractable flow problems.

The versatility of the finite element method in handling different kinds of flow boundaries and boundary conditions, aquifer anisotropy and heterogeneity has been demonstrated by many workers for steady flow complying with Darcy's law. (Zienkiewicz and Cheung, 1966; Taylor and Brown, 1967; Finn, 1967). Neuman and Witherspoon (1968, 1969, 1970, 1971) are among the first workers to apply the method to transient flow towards a well pumped at either constant discharge or constant drawdown. They extended the variational principles to cover both transient confined and unconfined flows, and solved a number of complex problems of flow through multi-layered aquifers. The usefulness and validity of the finite element approach was demonstrated by comparing the results with the few known analytical solutions. The flow in the entire region of the well-aquifer system was assumed to be in accordance with Darcy's law. No attempt was made to examine the localised non-Darcy behaviour which may exist if the well is heavily pumped or the well is packed with gravel material. Trollope, Stark and Volker (1970) were among the first to attempt to solve non-Darcy well flow. These workers derived a non-linear field equation describing steady two-dimensional flow through isotropic aquifers with the hydraulic gradient and the flow velocity as the dependent variables. The finite element analysis they used took no account of the two regime nature of the flow. The same non-linear equation was applied throughout the entire flow region. While this was the case for the flow they simulated in the laboratory, it may not be the case generally observed in the field.

In order to bring into focus the localised nature of the non-Darcy zone surrounding the well, the non-linear field equation should be applied only up to a point where the Reynolds number exceeds a certain critical value as determined from permeability tests on the aquifer sample.

The general problem of transient two regime well flow still remains to be solved. McCorquodale (1970) dealt with transient non-Darcy flow associated with rock-fill structure subjected to wave action, but the field equation he used differs from that describing the non-Darcy zone near a well. Problems of transient flow through aquifers towards wells are associated with elastic storage effects, which have to be taken into account in the continuity equation of flow, and in the case of unconfined aquifers, the movement of the water table must also be considered.

1.2 Scope of the Present Study

The purpose of the present section is to present theoretical and numerical methods for analysis of complex flow through aquifers towards wells.

The theory and fundamental principles of well hydraulics based on Darcy's law are reviewed and extended to describe non-Darcy flow which may occur in the close vicinity of the well. Generalised field equations and variational principles applicable for transient three-dimensional flow are developed. Energy approach to the two-regime flow problem is presented and an energy theorem is stated and proved. A powerful numerical method so called "finite element technique" is described and formulated to solve confined flow cases. The flexibility and usefulness of this method are demonstrated in the next section.

The method can be extended to the more complicated problems of transient free surface flow.

2. Basic Principles and Field Equations of Well Flow

2.1 Introduction

The flow through aquifers toward pumped wells is generally associated with two flow regimes. The first flow regime, referred to as 'Darcy flow regime', is in the main portion of the aquifer where the flow complies with Darcy's law. The second flow regime occurs in the immediate vicinity of the well where Darcy's law may be invalidated.

The hydraulic principles of well flow that have been outlined in the literature are based on the assumption that the flow remains laminar and obeys Darcy's law right to the face of the well boundary. In order to analyse the general problem of two regime well flow, it is necessary to extend these principles and develop generalised field equations which can describe both Darcy and non-Darcy flow.

The hydraulic principles and field equations developed herein are generally applicable to transient, three dimensional, two regime well flow. In this development, it is assumed that the two flow regimes, namely Darcy and non-Darcy are distinct, and that the Forchheimer non-linear velocity-gradient relation may be used to describe non-Darcy flow. The concept of Reynolds' number of flow is introduced, and a critical Reynolds number is used to express the transition between the two flow regimes. Tensor subscript notation is employed in the derivation of the field equations to describe flow through anisotropic aquifers.

2.2 Darcy's Law

2.2.1 Differential Form

According to Darcy's law, the macroscopic flow velocity is proportional to the hydraulic gradient taken in the flow direction. The constant of proportionality is termed "coefficient of hydraulic conductivity", and is observed to be dependent on the properties of groundwater as well as the characteristics of the aquifer medium. Among the various factors influencing this coefficient are grain size distribution, packing and shape of granular particles, temperature and chemical composition of groundwater.

The following generalisations are now introduced in order that Darcy's law may be written in its differential form which is applicable to a general problem of three dimensional flow through anisotropic and non-homogeneous aquifers.

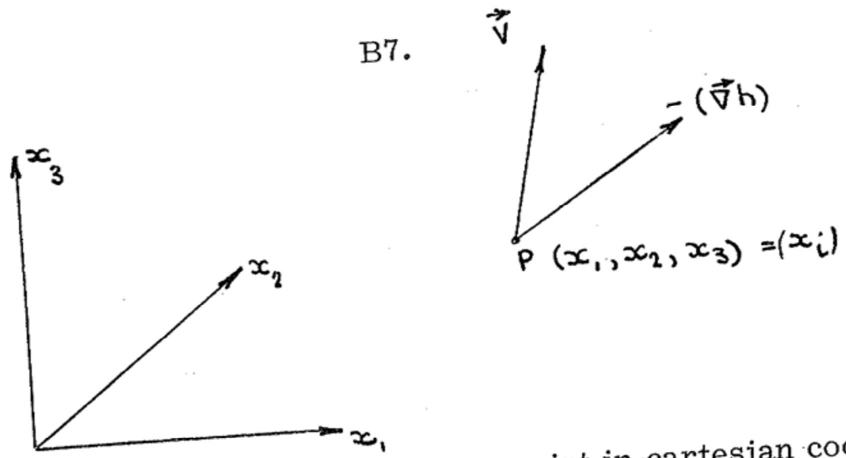

Fig. (2-1): Velocity and gradient at a point in cartesian coordinate system.

A right handed system of cartesian coordinate axes (x_1, x_2, x_3) with axis x_3 pointing vertically upward and plane $x_1 - x_2$ corresponding to a datum plane is adopted as shown in Fig. (2-1).

The hydraulic head $h(x_i, t)$ at point $P(x_1, x_2, x_3)$ is defined as the sum of the pressure head and the elevation of the point above plane $x_1 - x_2$. Thus the head may be expressed as

$$h(x_i, t) = p/\gamma + x_3 \tag{2-1}$$

where p is the hydrostatic pressure at the point, γ is the specific weight of water, and x_3 is the elevation of the point above datum plane $x_1 - x_2$.

Let v_1, v_2, v_3 be the three components of the velocity vector, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial x_2}$, $\frac{\partial h}{\partial x_3}$ be the three components of the hydraulic gradient, and $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be the three unit vectors along x_1, x_2 and x_3 axes respectively. The velocity vector, $\vec{V}$ and the hydraulic gradient vector, $\vec{\nabla} h$ may now be expressed as

$$\vec{V} = v_i \vec{e}_i \tag{2-2}$$

$$\vec{\nabla} h = \frac{\partial h}{\partial x_i} \vec{e}_i \tag{2-3}$$

where the repeated subscripts denote summation over the full range, from 1 to 3.

Thus for three dimensional flow through anisotropic aquifers, the general vector differential form of Darcy's law is given by

$$V = -\vec{\vec{K}} \; \vec{\nabla} h \tag{2-4}$$

Where $\vec{\vec{K}}$ denotes the hydraulic conductivity tensor, a second order symmetric tensor which may be expressed as

$$\vec{\vec{K}} = K_{ij} \vec{e}_i \vec{e}_j \qquad (2\text{-}5)$$

in which K_{ij} are the nine components of $\vec{\vec{K}}$, six of which are independent.

Equation (2-4) may also be written in the following tensor subscript form

$$v_i = - K_{ij} \frac{\partial h}{\partial x_j} \qquad (2\text{-}6)$$

If the aquifer is isotropic, the tensor $\vec{\vec{K}}$ will have one independent component, which is the coefficient of hydraulic conductivity. It follows that

$$K_{ij} = \delta_{ij} K \qquad (2\text{-}7)$$

where δ_{ij} denotes the Kronecker delta

On substituting equation (2-7) into equation (2-6), the following equations for Darcy flow through isotropic aquifers are obtained.

$$v_i = -K \delta_{ij} \frac{\partial h}{\partial x_j} \qquad (2\text{-}8a)$$

Contracting subscript j gives

$$v_i = -K \frac{\partial h}{\partial x_i} \qquad (2\text{-}8b)$$

In general K_{ij} and K will be functions of coordinates, unless the aquifer is homogeneous.

2.2.2 Range of Validity of Darcy's Law

The linear velocity-gradient relation dictated by Darcy's law may be derived theoretically by applying the Navier-Stokes differential equations of motion to the microscopic flow through the pore space of the aquifer medium, and integrating these equations over a macroscopie region (Sunada, 1965; Stark and Volker, 1967). In the derivation, the microscopic flow is assumed to be predominantly viscous in order that the inertial terms in the Navier-Stokes equations may be neglected. Experimental investigations have confirmed that the linear relationship ceases to be valid as the inertial terms become more important or turbulent flow develops.

By analogy to flow through pipes, the Reynolds number may be employed as an index to classify the flow into linear and non-linear flow regimes.

When adapted to flow through porous media, the Reynolds number is given by

$$\mathbb{R} = \frac{V\bar{d}\rho}{\mu} \qquad (2\text{-}9)$$

where ρ is the fluid density, V is the macroscopic bulk velocity, $\bar{d}$ is the characteristic grain diameter, and μ is the dynamic viscosity of the fluid. The characteristic grain diameter suggested by Hazen (1893) is the diameter of sand grain such that 10% by weight of the sample is of smaller size.

The above Reynolds number is not a completely satisfactory criterion to define departure from Darcy's law or to determine the onset of non-Darcy flow, mainly because it does not take sufficient account of the general shape of separate grains and the packing of these grains. Research is still needed to develop a better understanding of the flow transition. At present it is not possible to make reliable predictions of the validity limit of the linear relation for a given porous medium. Permeability tests on natural and artificial sands have shown that departure from the linear Darcy law appears when $\mathbb{R}$ reaches a range between about 1 and 10, and that the non-linear behaviour becomes evident at a value of $\mathbb{R}$ many times less than that corresponding to the onset of turbulence.

In view of the absence of a more satisfactory criterion, a critical value of the Reynolds number, $\mathbb{R}_{cr}$, has been used in the present study. The critical Reynolds number is defined as the limiting value above which the velocity-gradient relation is non-linear.

2.3 Equations for Non-Darcy Flow in the Vicinity of Wells

The non-Darcy flow in the vicinity of a pumped well is described by the non-linear Forchheimer relation, which for one-dimensional parallel flow may be written as

$$i = aV + bV^2 \qquad (2\text{-}10)$$

where i is the absolute hydraulic gradient, V is the absolute macroscopic velocity, a and b are the linear and non-linear coefficients of hydraulic resistance respectively.

Equation (2-10) has been derived theoretically via microscopic approach for both inertial laminar flow and turbulent flow (Stark and Volker, 1967). The nonlinear term in the equation has been shown to be caused by the increasing influence of the inertial forces in the case of inertial laminar flow, and by inertial and turbulent effects in the case of turbulent flow.

Permeability tests on natural and artificial porous media have confirmed that the equation may be used to describe the flow over a wide range of Reynolds number with the two coefficients of hydraulic resistance, a and b, remaining approximately constant (Sunada, 1965; Stark and Volker, 1967).

In order to describe three dimensional non-Darcy flow through anisotropic and non-homogeneous media, it is necessary to transform equation (2-10) into the following vector differential equation

$$\vec{\nabla} h = - (\vec{a} + \vec{b} \, |V|) \, \vec{V} \qquad (2\text{-}11)$$

where $\vec{a}$ and $\vec{b}$ are the two hydraulic resistance tensors, the components of which are a_{ij} and b_{ij} respectively.

Equation (2-11) may also be written in the following tensor subscript form.

$$\frac{\partial h}{\partial x_i} = -(a_{ij} + b_{ij} \, |V|) \, v_j \qquad (2\text{-}12)$$

where $|V|$ is the length of the velocity vector

$$|V| = (v_i \, v_i)^{\frac{1}{2}} \qquad (2\text{-}13)$$

The components of the effective hydraulic conductivity tensor, E_{ij}, are now defined in accordance with

$$E_{ij} = (a_{ij} + b_{ij} \, |V| \,)^{-1} \qquad (2\text{-}14)$$

where E_{ij} are functions of $|V|$.

For isotropic aquifers, equation (2-12) reduces to

$$\frac{\partial h}{\partial x_i} = - (a + b \, |V|) \quad v_i \qquad (2\text{-}15)$$

and E_{ij} becomes

$$E = (a + b \, |V| \,)^{-1} \qquad (2\text{-}16)$$

where E is termed "the coefficient of effective hydraulic conductivity".

2.4 Elastic Storage of Aquifers

2.4.1 Confined Aquifers

The concept of volume elasticity of confined aquifers has long been established by laboratory and field observations. Phenomena such as fluctuations of water levels in wells in response to barometric pressure changes, earthquake effects and tidal fluctuations constitute good evidence of the compressibility of these aquifers.

Transient flow under confined conditions is due to the volume of water released from the aquifer storage because of water expansion and aquifer compression in response to the decline in pressure head caused by pumping. If it is assumed that the stress-strain relationship of the aquifer complies with Hooke's law, then an equation relating the decrease in hydraulic head and the released volume of water may be derived as follows:-

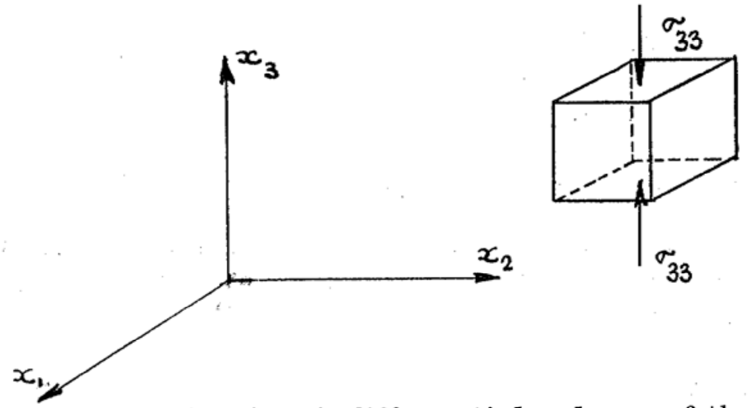

Fig. (2.2): A differential volume of the aquifer.

Consider an elemental volume $\delta\forall$ in a compressible aquifer medium as shown in Fig. (2.2), it follows that

$$\delta\forall = (\delta x_1 \delta x_2)\,\delta x_3$$
$$= \delta A\,\delta z$$

where δA is the incremental horizontal area, and δz is the incremental height.

If it is assumed that the water pressure acts throughout $\delta\forall$, the compressive stress of the skeleton acts over δA, the lateral deformations of the volume are negligible, and the atmospheric pressure is constant, then the following equation may be written provided that the arching action of the overlying material is negligible.

$$p + \sigma_{33} = \text{constant}$$

where p is the water pressure and σ_{33} is the vertical compressive stress.

Differentiating equation (2-17) leads to

$$d\sigma_{33} = -dp \tag{2-17}$$

The stress-strain relationship for the compressible aquifer medium is given by

$$d\,\frac{(\delta z)}{\delta z} = -\alpha\, d\sigma_{33} \tag{2-18}$$

where α is the vertical compressibility of the aquifer medium.

The volume of solid material δV_s contained in δV is given by

$$\delta V_s = (1-n)\,\delta V$$
$$= (1-n)\,\delta A\,\delta z$$

where n is the effective porosity of the aquifer.

As the compressibility of the individual grains of the solid skeleton is small compared to the change in porosity, n, the solid volume may be assumed constant. Hence it follows that

$$d(\delta V_s) = 0$$
$$\frac{d(\delta V_s)}{\delta A} = \delta z\, d(1-n) + (1-n)\, d(\delta z)$$
$$dn = (1-n)\,\frac{d(\delta z)}{\delta z} \tag{2-19}$$

Equations (2-17) and (2-18) may be substituted into equation (2-19) to give

$$dn = -(1-n)\alpha\, d\sigma_{33}$$
$$dn = \alpha\,(1-n)dp \tag{2-20}$$

The change in density of water and the differential pressure are related by

$$d\rho = \beta\rho_o\, dp \tag{2-21}$$

where β and ρ_o are the volume compressibility and a reference density of water respectively.

The mass of water contained in δV is

$$\delta M = \rho\, n\, \delta z\, \delta A$$

On taking differentials

$$d(\delta M) = \delta A\left(\rho n\, d(\delta z) + \rho\, \delta z\, dn + \delta z\, n\, d\rho\right)$$

$$\frac{d(\delta M)}{\delta \forall} = \rho n \frac{d(\delta z)}{\delta z} + \rho\, dn + n\, d\rho \qquad (2\text{-}22)$$

Equations (2-18), (2-20) and (2-21) may be substituted into Equation (2-22) to give

$$\frac{d(\delta M)}{\delta \forall} = (\rho\alpha + n\beta\rho_o)\, dp$$

in which the density ρ may be approximated by ρ_o

Dividing by ρ gives

$$\frac{d(\delta M)}{\rho\,\delta \forall} = (\alpha + n\beta)\, dp \qquad (2\text{-}23)$$

The hydraulic head has been defined in accordance with

$$h = \frac{p}{\gamma} + x_3$$

where x_3 is kept constant

Differentiating gives

$$dp = \gamma\, dh \qquad (2\text{-}24)$$

Substituting equation (2-24) into equation (2-23)

$$\frac{d(\delta M)}{\rho\,\delta \forall} = \gamma(\alpha + n\beta)\, dh \qquad (2\text{-}25a)$$

If the coefficient S_s is defined such that

$$S_s = \gamma(\alpha + n\beta)$$

then substitution into equation (2-25a) leads to the required expression

$$\frac{d(\delta M)}{\rho\,\delta \forall} = S_s\, dh \qquad (2\text{-}25b)$$

The coefficient S_s of dimension L^{-1} has been termed the 'specific storage' of the aquifer, and may be defined as the volume of water released from storage in a unit volume of an aquifer due to compression of the aquifer and expansion of water under a unit decline in the average head within the unit volume.

In the case of confined aquifers of constant thickness it is common to refer to the coefficient of storage, S, which is defined in accordance with

$$S = S_s m \qquad (2\text{-}26)$$

where m is the saturated thickness of the confined aquifer.

2.4.2 Unconfined Aquifers

The released volume of water from unconfined aquifers in response to the reduction in head results mostly from the dewatering of the zone through which the water table moves. The effect of aquifer compression and expansion of water in the saturated region below the water table, if it is appreciable, is only noticed shortly after the start of pumping when the delayed gravity drainage is almost absent. After this initial period, the gravity drainage due to specific yield becomes appreciable and increases at a diminishing rate with time of pumping. The manner in which specific yield varies with time is not yet clearly understood. Additional field and laboratory research is needed to develop a better understanding. Boulton (1963) suggests that delay yield is an exponential function of time and is proportional to the drawdown of the water table.

Following Boulton (1963), the general expression for the coefficient of storage in an unconfined aquifer may be written as -

$$S = \epsilon + mS_s \qquad (2\text{-}27)$$

where ϵ is the specific yield of the unconfined aquifer and m is the saturated thickness. For all practical purposes S may be approximated by

$$S \doteq \epsilon \qquad (2\text{-}27)$$

The expression for specific yield at time t from the start of pumping is given by

$$\epsilon = D\epsilon \int_0^t e^{-D(t-\tau)} d\tau \qquad (2.28)$$

where D is termed the delay yield index.

2.5 Derivation of Generalised Field Equations

2.5.1 The Continuity Equation of Flow

The continuity equation of flow through aquifers may be developed by applying the law of conservation of matter, according to which the net rate of mass entering the closed boundary of an arbitrary volume situated in the flow field is balanced by the rate of accumulation of mass within the volume.

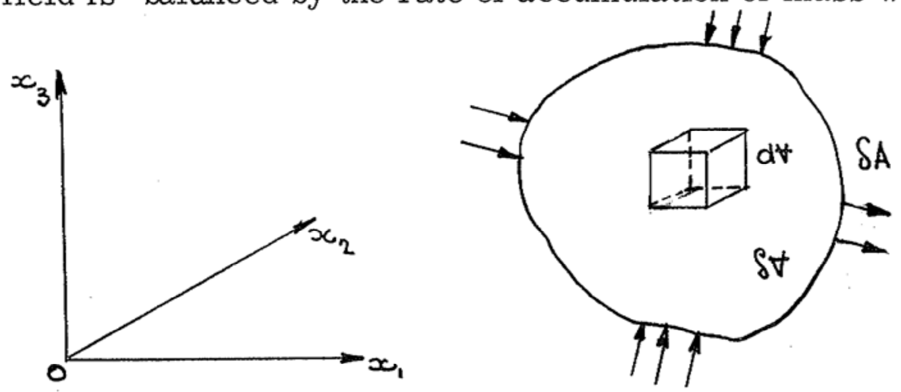

Fig. (2-3): An arbitrary closed region in the flow field.

Consider an elemental volume $\delta\forall$ of an aquifer situated in the flow field as shown in Fig. 2-3. Let δA be the surface area of the closed boundary of $\delta\forall$.

The net rate of mass entering δA is given by

$$-\int_{\delta A} n_i \rho v_i \, dA$$

where n_i are the components of the unit outward normal vector of the differential area dA, ρ is the density of water, and v_i are the components of the velocity vector.

The rate of mass accumulated within $\delta\forall$ is

$$\int_{\delta\forall} \frac{\partial M}{\partial t} \frac{d\forall}{\delta\forall}$$

Since mass is conserved, it follows that

$$-\int_{\delta A} n_i \rho v_i \, dA = \int_{\delta\forall} \frac{\partial M}{\partial t} \frac{d\forall}{\delta\forall} \qquad (2\text{-}29)$$

The divergence theorem may now be applied to transform the surface integral into a volume integral. On replacing the left-hand term of equation (2-29) by its equivalent volume integral, and rearranging the terms, the following equation results

$$\int_{\delta\forall} \left[\rho \frac{\partial v_i}{\partial x_i} + \frac{\partial M/\partial t}{\delta\forall} \right] d\forall = 0 \qquad (2\text{-}30)$$

Since the choice of $\delta\forall$ has been made arbitrarily, the integrand in equation (2-30) may be shown to vanish identically. It follows that

$$\frac{\partial v_i}{\partial x_i} = \frac{-\partial M/\partial t}{\rho \delta \forall} \tag{2-31}$$

The rate of mass accumulated and the rate of change of hydraulic head are related by

$$\frac{-\partial M/\partial t}{\rho \delta \forall} = S_s \frac{\partial h}{\partial t}$$

which may be substituted into equation (2-31) to give the required continuity equation.

$$\frac{\partial v_i}{\partial x_i} = - S_s \frac{\partial h}{\partial t} \tag{2-32}$$

2.5.2 Differential Equations of Motion

The Darcy and the Forchheimer differential equations of flow have been derived previously. The linear Darcy equations are applicable within the region R^D in the outer zone of the aquifer. The non-linear Forchheimer equations are applicable within the region R^N in the immediate vicinity of the well.

To determine whether a point in the flow field belongs to R^D or R^N, the critical Reynolds number is used as a criterion. A point in the flow field belongs to R^N if and only if its Reynolds number is greater than the critical Reynolds number, otherwise it belongs to R^D.

The equations of motion may now be rewritten as

$$v_i = - K_{ij} \frac{\partial h}{\partial x_j} \qquad \text{for } \mathbb{R} \leqslant \mathbb{R}_{cr}$$

and

$$\frac{\partial h}{\partial x_i} = -(a_{ij} + b_{ij}|V|)\, v_j \qquad \text{for } \mathbb{R} > \mathbb{R}_{cr}$$

where $\mathbb{R}_{cr}$ is the critical Reynolds number.

2.5.3 The Generalised Field Equations

(i) Darcy Flow

The Darcy differential equations and the continuity equation may be combined together to give the following second order linear field equation, which is generally applicable to transient three-dimensional Darcy flow in anisotropic and non-homogeneous aquifers.

$$\frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial h}{\partial x_j}\right) = S_s \frac{\partial h}{\partial t} \tag{2-33}$$

For isotropic and homogeneous aquifers, the above field equation reduces to

$$\frac{\partial}{\partial x_i}\left(K \frac{\partial h}{\partial x_i}\right) = S_s \frac{\partial h}{\partial t} \tag{2-34}$$

in which the coefficient K is a constant. Equation (2-34) now becomes

$$\frac{\partial^2 h}{\partial x_i \, \partial x_i} = \frac{S_s}{K} \cdot \frac{\partial h}{\partial t} \tag{2-35a}$$

If the coefficient of diffusivity of aquifers, ν , is defined as

$$\nu = \frac{K}{S_s}$$

equation (2-35a) may be written in the form

$$\frac{\partial^2 h}{\partial x_i \, \partial x_i} = \frac{1}{\nu} \frac{\partial h}{\partial t} \tag{2-35b}$$

Equation (2-35b) has been solved in closed forms for a number of cases of axi-symmetric well flow. Several methods for evaluating the aquifer properties based on the mathematical solutions are available in the literature (Huyakorn and Dudgeon, 1972).

(ii) Non-Darcy Flow

The non-linear field equation describing non-Darcy flow in anisotropic aquifers may be obtained by combining the Forchheimer differential equations of motion with the continuity equation.

From equation (2-12), it follows that

$$v_j = -(a_{ij} + b_{ij} |V|)^{-1} \frac{\partial h}{\partial x_i} \tag{2-36}$$

which may be substituted into equation (2-32) to result in

$$\frac{\partial}{\partial x_j}\left[(a_{ij} + b_{ij}|V|)^{-1} \frac{\partial h}{\partial x_i}\right] = S_s \frac{\partial h}{\partial t} \tag{2-37}$$

The detailed analysis of equation (2-37) is beyond the scope of the present study. The present analysis assumes that the aquifer is isotropic. Such an assumption leads to a simplified field equation involving only h as a dependent variable.

For isotropic aquifers, the Forchheimer differential equations reduce to

$$\frac{\partial h}{\partial x_i} = -(a + b\ |V|)\ v_i \qquad (2\text{-}38)$$

Contracting subscript i in equation (2-38) gives

$$\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_i} = (a+b\ |V|)^2\ v_i\ v_i \qquad (2\text{-}39)$$

The absolute hydraulic gradient is defined as

$$\left|\frac{\partial h}{\partial l}\right| = \left(\frac{\partial h}{\partial x_i}\cdot\frac{\partial h}{\partial x_i}\right)^{1/2} \qquad (2\text{-}40a)$$

which may be rearranged to give

$$\frac{\partial h}{\partial x_i}\cdot\frac{\partial h}{\partial x_i} = \left|\frac{\partial h}{\partial l}\right|^2 \qquad (2\text{-}40b)$$

Now

$$v_i\ v_i = |V|^2 \qquad (2\text{-}40c)$$

Substituting equations (2-40b) and (2-40c) into equation (2-39) gives

$$\left|\frac{\partial h}{\partial l}\right|^2 = (a + b\,|V|)^2\ |V|^2$$

which may be rewritten as

$$\left|\frac{\partial h}{\partial l}\right| = (a+b\ |V|)\ \ |V| \qquad (2\text{-}41)$$

Solving for $|V|$

$$|V| = -\frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{|\partial h/\partial l|}{b}} \qquad (2\text{-}42)$$

From equation (2-41), it follows that

$$\frac{|V|}{|\partial h/\partial l|} = \frac{1}{a + b|V|} \qquad (2\text{-}43)$$

Combining equations (2-38), (2-42) and (2-43) leads to

$$v_i = \frac{\left(-\frac{\partial h}{\partial x_i}\right)}{\left|\frac{\partial h}{\partial l}\right|}\left[-\frac{a}{2b} + \sqrt{\left(+\frac{a}{2b}\right)^2 + \frac{|\partial h/\partial l|}{b}}\ \right] \qquad (2\text{-}44)$$

Equation (2-44) may now be substituted into the continuity equation to give the required field equation.

Thus

$$\frac{\partial}{\partial x_i}\left[\left(-\frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{|\partial h/\partial l|}{b}}\ \right)\left(\frac{-\partial h/\partial x_i}{(|\partial h/\partial l|)}\right)\right] = S_s\,\frac{\partial h}{\partial t} \qquad (2\text{-}45)$$

The above field equation is used to describe non-Darcy flow in the vicinity of pumped wells.

2.6 Initial and Boundary Conditions

Initial and boundary conditions refer to conditions of a well-aquifer system at a particular time before pumping starts, and conditions prevailing on the boundaries of the system respectively. Solutions of the previously derived field equations satisfying these conditions can be used to describe a given flow system. On small scales of space and time, the actual variations of boundary and initial conditions are generally irregular. However, if the process of averaging in time and space is introduced, the averaged initial and boundary conditions become regular and may be assigned mathematical expressions.

(i) Initial Conditions

The initial distribution of hydraulic head throughout the aquifer region is assumed to be a prescribed function of coordinates. If the function is continuous, the correct solution of the problem should tend to this function as the time approaches zero. If, on the other hand, the initial distribution is discontinuous at points or on surfaces, the discontinuities must disappear after a very short time and the solution must converge to the initial distribution at all points where this distribution is continuous.

The distribution of head of a well-aquifer system initially at rest usually corresponds to the height of water table above the datum plane, and the initial condition may be expressed as

$$h(x_i, 0) = h^o(x_i) \qquad x_i \in \bar{R} \tag{2-46}$$

where $h^o(x_i)$ is the prescribed head function, and $\bar{R}$ is the closed region of the flow system.

(ii) Boundaries and Boundary Conditions

The boundaries of a well-aquifer system and the prevailing conditions that are of common occurrence are classified as follows:-

(a) Pervious Boundaries

Pervious boundaries of a flow system are defined as boundaries or portions of the boundary across which there can be flux interchanges between the flow system and its surroundings.

Two common examples of boundaries of this kind are screen sections along the vertical axis of a well, and portions of the aquifer boundary intercepted by recharge sources.

The conditions prevailing on these boundaries are as follows:-

Type 1: Prescribed Flux or Prescribed Flow Rate

If the flux distribution on the boundary is known at any instant of time, the resulting boundary condition will be referred to as "prescribed flux condition". If, on the other hand, the total flow rate across the boundary is a known function of time, the resulting boundary condition will be referred to as "prescribed flow rate condition".

Both the prescribed flux condition and prescribed flow rate condition will be referred to as "Type 1 boundary condition". They may be expressed mathematically in the following manner.

Let B_1^p be the portion of the pervious boundary where the flux distribution or the flow rate is prescribed. If $\bar{q}$ denotes the prescribed inflow flux, then the prescribed flux condition may be written as

$$v_i n_i = -\bar{q}\ (x_i,\ t) \qquad (x_i) \in B_1^p \tag{2-47}$$

where v_i are the components of the velocity vector $\vec{V}$, and n_i are the components of the outward normal vector $\vec{n}$ of the boundary surface.

Also if $\bar{Q}(t)$ denotes the prescribed flow rate at time t, the prescribed flow rate condition may be written as

$$Q = \bar{Q}(t) \qquad \text{across } B_1^p \tag{2-48}$$

Two common examples of type 1 boundary conditions are the prescribed flux condition on the leaky boundary of an aquifer and the prescribed flow rate condition prevailing on the boundary of a well operating at a known discharge.

Type 2: Prescribed Head Distribution

If the distribution of hydraulic head on the flow boundary is known at any instant of time, the resulting boundary condition is referred to as the condition of "prescribed head distribution".

A common example is where the aquifer boundary is intercepted by a very large body of surface water. The water level at this location is uniform and unaffected by pumping. Another example is the condition of constant hydraulic head on wetted screened sections in a well operating at a constant water level.

The mathematical expression of this type of boundary condition is given by

$$h = \bar{h}(x_i, t) \quad \text{on } B_2^p \tag{2-49}$$

where $\bar{h}$ is the prescribed head function, and B_2^p denotes the portion of the pervious boundary on which the head is prescribed.

(b) Impervious Boundaries

Across impervious boundaries of a flow system, no flux exchange between the system and its surroundings is possible. Accordingly, the velocity component normal to the boundary is zero.

Typical examples of boundaries of this kind are rock beds overlying and underlying the aquifers, and cased sections of the well boundary. The prevailing boundary condition may be expressed as -

$$v_i n_i = 0 \quad \text{on } B^c \tag{2-50}$$

where B^c denotes the impervious boundary.

(c) Free Surfaces

A free surface may be defined as a stream surface along which the pressure is uniform. In problems of flow towards water table wells in which the effects of capillary fringe on flow in the saturated region may be neglected, the water table may be taken as the upper bounding free surface of the flow, and the pressure on it may be considered uniform and equal to the atmospheric pressure. The position of the free surface at any instant of time during pumping is unknown and has to be located by trial and error during the course of solution of the flow problem.

Two conditions have to be satisfied on the free surface. The first is

$$h(x_i, t) = x_3 = z(x_1, x_2, t) \quad \text{on } B^F \tag{2-51}$$

where $z(x_1, x_2, t)$ is the vertical height of the free surface at (x_1, x_2, t) above the datum plane, and B^F denotes the free surface boundary.

The second condition is the requirement of continuity of flow across the free surface

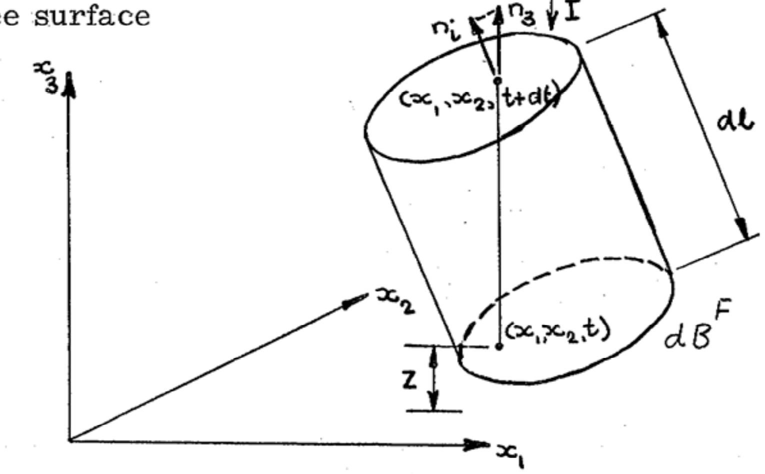

Fig. (2-4): A differential element of the free surface.

To derive this condition, consider a differential element of the free surface, dB^F as shown in Fig. (2-4).

Let $z(x_1, x_2, t)$ denote the elevation at time t of a point (x_1, x_2) on dB^F above the datum plane. At later time t+dt point (x_1, x_2, t) is moved to $(x_1, x_2, t+dt)$ due to a shift in the free surface. If the net average of vertical infiltration into the free surface is I, and the specific yield capacity is ϵ then the volume of water contained in the elemental volume, $d\forall = dB^F d\ell$, may be expressed as

$$\epsilon \, dB^F d\ell \quad \text{or} \quad \epsilon \, dB^F \frac{\partial z}{\partial t} n_3 \, dt$$

where n_3 is the vertical component of the normal vector of dB^F.

This quantity is balanced by the net volume of inflow during an incremental time dt, given by

$$(v_i n_i + I n_3) \; dB^F dt$$

Hence

$$\epsilon \, dB^F \frac{\partial z}{\partial t} n_3 \; dt = (v_i n_i + I n_3) \, dB^F dt$$

which may be rearranged to result in the required boundary condition.

$$v_i n_i = (I - \epsilon \frac{\partial z}{\partial t}) \, n_3 \qquad \text{on } B^F \tag{2-52}$$

Since the choice of dB^F has been made arbitrarily, equation (2-52) is applicable on the entire free surface B^F.

If the flow is steady, the right hand term of equation (2-52) vanishes and the following condition results

$$v_i n_i = 0 \quad \text{on } B^F \tag{2-53}$$

(d) Seepage Faces

Unconfined flow problems are complicated by the presence of free surfaces and the presence of exit flow boundaries directly connected to these surfaces. These exit flow boundaries are referred to as "seepage faces"; their length or extent is unknown, but on them the pressure is uniform and, in most cases, may be taken to be atmospheric. Seepage takes place across the faces resulting in the seepage flow, which may be a significant proportion of the total flow.

The seepage face of a water-table well is the vertical drainage face located directly below the water table and above the pumping water level. The prevailing boundary condition may be written as

$$h(x_i, t) = x_3 \quad \text{on } B^S \tag{2.54}$$

where B^S denotes the seepage face surface.

(e) Aquifer Interfaces

Frequently in practice, a well penetrates a number of aquifers having uniform but different hydraulic properties. The interface over which the change in hydraulic properties takes place may be idealized as a surface forming a common boundary between the two adjacent aquifers. Each aquifer is treated as a closed subregion, and the flow in the subregion is connected to the flow in the adjacent subregion by the flux crossing the common interface. If the head distribution in two such aquifers is designated by h_1 and h_2 and the velocity components by v_{i1} and v_{i2}, then the boundary conditions on the interface may be expressed as

$$h_1 = h_2 \quad \text{on } B^I \tag{2-55a}$$

and

$$v_{i1} n_i = v_{i2} n_i \quad \text{on } B^I \tag{2-55b}$$

where B^I denotes the interface boundary surface, and n_i are the components of a unit vector normal to the surface.

3. Variational Principle for Two Regime Well Flow

3.1 Introduction

In the previous chapter the fundamental approach to the general problem of two regime well flow was presented, in which the flow was described by two field equations and the problem was reduced to that of finding a function satisfying these equations as well as the initial and boundary conditions.

An alternative approach is possible via variational methods. In this approach an extremum principle valid over the entire flow region is postulated. The required solution is the one extremising a certain quantity Ω , termed 'functional', subject to the same conditions of the flow system. The functional is defined by suitable integration of the unknown quantities over the region.

While the two approaches are mathematically equivalent in the sense that an exact solution of one being the solution of the other, the variational approach is particularly useful for the approximate computation of the solution by the finite element method. Furthermore the governing field equations may be obtained from the necessary conditions for extremisation of the functional.

The variational principle for steady state Darcy flow through aquifers was developed by Mauersberger (1965) and later extended by Neuman and Witherspoon (1970), (1971) to transient flow. The principle for non-Darcy flow has not been fully developed. Only the case of steady state two-dimensional flow was treated by Volker (1969) and McCorquodale (1969).

The purpose of this chapter is to present a generalised variational principle applicable to transient two-regime well flow through confined and unconfined aquifers. The energy theorem describing the flow is established and directly related to the variational principle. Via this theorem a physical meaning is assigned to the functional.

3.2 Development of Variational Principle

3.2.1 Variational Forms of the Field Equations

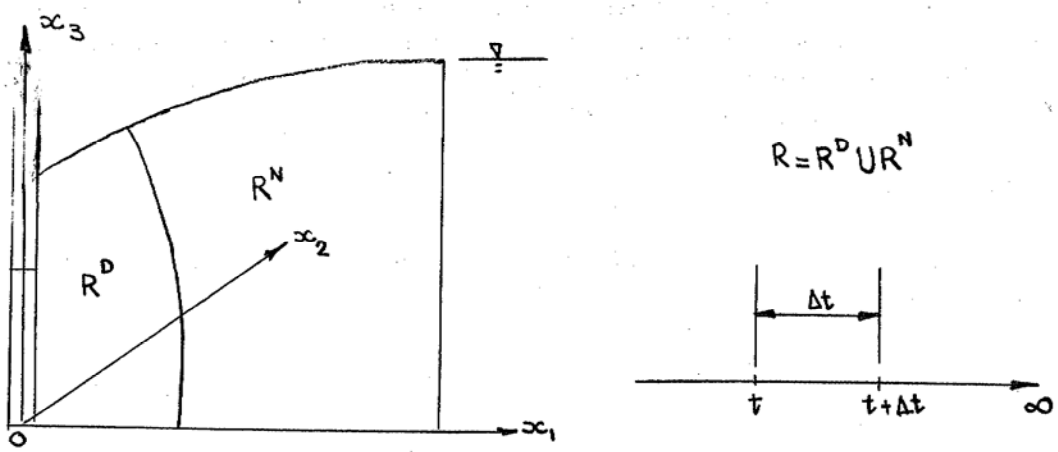

Fig. (3-1) 3-Dimensional space region and an open time domain.

Variational forms of the previously derived field equations may be obtained by considering an equivalent variational problem and employing the Euler-Lagrange equation from calculus of variations, (Wienstock, 1952).

Consider the general well-aquifer system shown in Fig. (3-1). As indicated above, (x_1, x_2, x_3) represents a right-handed system of cartesian coordinate axes, R^D and R^N are the Darcy and non-Darcy subregions of flow respectively.

Let $h\,(x_i, t)$ be an admissible function with second space and first time derivatives which are continuous everywhere in a given flow region R, and let the time domain be subdivided into a number of finite time increments.

Assuming that $h(x_i, t)$ is known at a particular time t, the general functional to be extremised over the space region R and the time increment Δt may be expressed as

$$\left[\Omega(h)\right]_R = \int_t^{t+\Delta t} \int_R G\left(h, \frac{\partial h}{\partial x_i}, \frac{\partial h}{\partial t}, x_i, t\right) dR\, dt \qquad (3\text{-}1)$$

The extremum problem is now reduced to seeking the function $h\,(x_i, t)$ which holds the above functional stationary. A necessary condition is the Euler-Lagrange equation, which may be written as

$$\frac{\partial G}{\partial h} - \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial\left(\frac{\partial h}{\partial x_i}\right)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial G}{\frac{\partial h}{\partial t}}\right) = 0 \qquad (3\text{-}2)$$

where the repeated subscripts represent summation over the full range, from 1 to 3.

Equation (3-2) represents various classes of partial differential equation. The previously derived field equations may be shown to belong to one of these classes. Thus on equating the field equations to equation (3-2), expressions for function G may be obtained.

(i) Non-Darcy Field Equation

The field equation describing non-Darcy flow through isotropic aquifers is now rewritten as

$$\frac{\partial}{\partial x_i}\left[\left(-\frac{a}{2b}+\sqrt{\left(\frac{a}{2b}\right)^2+\frac{\left|\frac{\partial h}{\partial l}\right|}{b}}\right)\left(-\frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|}\right)\right] = S_s\frac{\partial h}{\partial t} \qquad (3\text{-}3)$$

The above equation is applicable everywhere in the non-Darcy subregion R^N. On equating this equation to equation (3-2), the following equations result:

$$\frac{\partial G}{\partial h} = 0$$

$$\frac{\partial G}{\partial\left(\frac{\partial h}{\partial x_i}\right)} = \left(-\frac{a}{2b}+\sqrt{\left(\frac{a}{2b}\right)^2+\frac{\left|\frac{\partial h}{\partial l}\right|}{b}}\right)\left(-\frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|}\right)$$

$$\frac{\partial G}{\partial\left(\frac{\partial h}{\partial t}\right)} = h\,S_s$$

Integrating the above expressions leads to

$$G = -\frac{a}{2b}\left|\frac{\partial h}{\partial \ell}\right| + \frac{2b}{3}\left[\left(\frac{a}{2b}\right)^2+\frac{\left|\frac{\partial h}{\partial \ell}\right|}{b}\right]^{3/2} + S_s\,h\,\frac{\partial h}{\partial t} \qquad (3\text{-}4)$$

The required functional $[\Omega(h)]_{R^N}$ over subregion R^N is now expressible as

$$[\Omega(h)]_{R^N} = \int_t^{t+\Delta t}\int_{R^N}\left[-\frac{a}{2b}\left|\frac{\partial h}{\partial l}\right| + \frac{2b}{3}\left\{\left(\frac{a}{2b}\right)^2+\frac{\left|\frac{\partial h}{\partial l}\right|}{b}\right\}^{3/2} + S_s\,h\,\frac{\partial h}{\partial t}\right] dR\,dt \qquad (3\text{-}5)$$

(ii) Darcy Flow

The field equation describing Darcy flow through anisoptropic aquifers is rewritten as

$$\frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial h}{\partial x_j}\right) = S_s \frac{\partial h}{\partial t} \qquad (3\text{-}6)$$

Equation (3-6) is applicable everywhere in R^D. On equating this equation to equation (3-2), the following relations are obtained:

$$\frac{\partial G}{\partial h} = 0$$

$$\frac{\partial G}{\partial\left(\frac{\partial h}{\partial x_i}\right)} = K_{ij} \frac{\partial h}{\partial x_j}$$

$$\frac{\partial G}{\partial\left(\frac{\partial h}{\partial t}\right)} = S_s h$$

Integrating the above expressions leads to

$$G = \tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + S_s\, h \frac{\partial h}{\partial t} \qquad (3\text{-}7)$$

Hence the functional over subregion R^D is given by

$$[\Omega(h)]_{R^D} = \int_t^{t+\Delta t} \int_{R^D} \left[\tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + S_s\, h \frac{\partial h}{\partial t} \right] dRdt \qquad (3\text{-}8)$$

For isotropic aquifers, equation (3-8) reduces to

$$[\Omega(h)]_{R^D} = \int_t^{t+\Delta t} \int_{R^D} \left[\tfrac{1}{2} K \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} + S_s\, h \frac{\partial h}{\partial t} \right] dRdt \qquad (3\text{-}9)$$

(iii) Statement of the Variational Problem

Let R be the union of R^N and R^D. The functional $[\Omega(h)]_R$ over R is expressible as

$$[\Omega(h)]_R = [\Omega(h)]_{R^N} + [\Omega(h)]_{R^D} \qquad (3\text{-}10)$$

where $[\Omega(h)]_{R^N}$ and $[\Omega(h)]_{R^D}$ are the two portions contributed by R^N and R^D respectively. Their expressions are given in equations (3-5) and (3-8) respectively.

The variational problem reduces to finding an admissible function that extremises $[\Omega(h)]_R$ and also satisfies the existing initial and boundary conditions of the flow system. The classification of flow boundaries according to their physical nature has been presented previously.

3.2.2 Treatment of Initial and Boundary Conditions

(i) Initial Condition

At a particular time, taken as an initial time, the head distribution throughout the space region of the flow system is known. If, in the extremisation of the function, the time integration is carried out between time 0 and Δt, the admissible function will automatically satisfy the following initial condition.

$$h(x_i, 0) = h^o(x_i) \qquad (x_i) \in \bar{R}; \quad \bar{R} = RUB$$

where $h^o(x_i)$ is the initial head, $\bar{R}$ is the closed region of the flow system, R and B are the interior and the boundary of $\bar{R}$ respectively.

(ii) Boundary Conditions

In extremising the functional, the requirements of the conditions on the flow boundary must be met. These requirements correspond to extra terms that have to be added to the previously derived functional.

The additional terms only exist on the boundary and vanish elsewhere in the flow region.

(a) Prescribed Boundary Conditions

Except for the free surface, the conditions prevailing on portions of the entire flow boundary fall into one of the following types.

Type 1: Prescribed Flux Condition.

Let B_1 be the portion of B where the flux is prescribed. The prescribed flux condition may be rewritten as

$$v_i n_i = -\bar{q} \qquad \text{on } B_1$$

where $\bar{q}$ is the prescribed inflow flux, v_i and n_i are components of the velocity and the outward normal vectors.

The additional term on B_1 that has to be added to the functional $[\Omega(h)]_R$ is given by

$$\int_t^{t+\Delta t} \int_{B_1} h\,\bar{q}\; dBdt$$

Type 2: Prescribed Head Condition

Let B_2 be the portion of B where the head function is prescribed. The condition prevailing on B_2 may be rewritten as,

$$h(x_i, t) = \bar{h} \qquad (x_i) \in B_2$$

where $\bar{h}$ is the prescribed head function.

The existing term on B_2 that has to be added to the functional $[\Omega(h)]_R$ is

$$-\int_t^{t+\Delta t} \int_{B_2} (h - \bar{h})\, v_i\, n_i \; dBdt$$

However, if the admissible function $h(x_i, t)$ is chosen to be equal to $\bar{h}$ everywhere on B_2, the integrand of the above integral will vanish identically and the integral may be neglected.

(b) Free Surface Conditions

The two conditions prevailing on the free surface of flow toward water table wells have been derived earlier in this report. They may be rewritten as follows:-

Let B^F be the free surface portion of the entire flow boundary B.

The first condition that must be satisfied on B^F is

$$h(x_i, t) = z(x_1, x_2, t)$$

where z is the elevation of the free surface above datum plane $x_1 - x_2$.

The second condition is

$$v_i n_i = (I - \epsilon \frac{\partial z}{\partial t}) n_3$$

where I is the vertical infiltration into the free surface, and ϵ is the specific yield of the aquifer material.

The additional terms on B^F due to these two conditions are given by

$$- \int_t^{t+\Delta t} \int_{B^F} (h - z) \; v_i n_i \; dBdt$$

and

$$- \int_t^{t+\Delta t} \int_{B^F} \left[z \left(I - \epsilon \frac{\partial z}{\partial t} \right) n_3 \right] dBdt$$

respectively.

3.3 Energy Approach to Well Flow Problems

3.3.1 Dissipation of Energy in the Flow Region

The movement of groundwater occurs through the interconnected portion of the existing porespace within the aquifer medium. While flowing, the water particle loses some of its energy due to friction against the walls of granular particles along the seepage path. The loss of hydraulic energy per unit distance travelled is usually expressed in terms of the hydraulic gradient.

When the macroscopic flow velocity lies within the range of Darcy flow, it is observed to be linearly related to the hydraulic gradient. Departure from the linear relationship starts at velocities greater than a critical value corresponding to the transition from linear regime to non-linear regime. The non-linear relationship has been represented by the Forchheimer velocity-gradient relation.

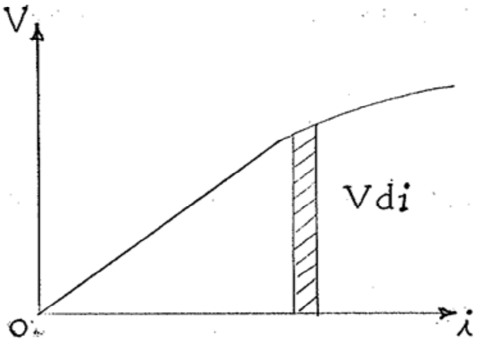

Fig. (3-2) Velocity-gradient relationship for a hypothetical aquifer.

A velocity-gradient curve for a hypothetical isotropic aquifer is shown in Fig. (3-2). As indicated above, the area of the shaded strip under the curve represents an increment of the rate of dissipation of hydraulic energy per unit weight of water. The total rate of energy dissipated within the aquifer volume may be evaluated as follows:

Consider an arbitrary volume R of the aquifer situated in the field of flow. Let a function $\emptyset$, termed 'dissipation function', be defined in accordance with

$$\dot{\emptyset} = \int V di \qquad (3\text{-}11)$$

$\dot{\emptyset}$ represents the rate of dissipation of hydraulic energy per unit weight of water. The total rate of energy dissipation within volume R of the aquifer medium, $\dot{\chi}$, is given by

$$\dot{\chi} = \gamma \int_R \dot{\emptyset}\, dR = \gamma \int_R \left(\int V di\right) dR \qquad (3\text{-}12)$$

where γ is the specific weight of water.

The integral in brackets may be integrated, if an expression relating V and i is given.

3.3.2 Dissipation Function for 3-Dimensional Flow

A general expression of the dissipation function for three dimensional flow is

$$\dot{\phi} = - \int v_i \; d\left(\frac{\partial h}{\partial x_i}\right) \qquad (3\text{-}13)$$

where the repeated subscripts represent summation over the full range, from one to three.

(i) Darcy Flow

For Darcy flow through anisotropic aquifers, the dissipation function becomes

$$\dot{\phi} = \int K_{ij} \frac{\partial h}{\partial x_j} \; d\left(\frac{\partial h}{\partial x_i}\right)$$

$$\dot{\phi} = \tfrac{1}{2} \; K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + \text{constant} \qquad (3\text{-}14)$$

(ii) Non-Darcy Flow

The dissipation function for non-Darcy flow obeying the Forchheimer velocity-gradient relation is given by

$$\dot{\phi} = \int (a_{ij} + b_{ij} \; |V|)^{-1} \frac{\partial h}{\partial x_j} d\left(\frac{\partial h}{\partial x_i}\right) \qquad (3\text{-}15)$$

where the integration may not be readily carried out as $|V|$ is also a function of the hydraulic gradient.

However, if the aquifer is isotropic, the integrated expression for $\dot{\phi}$ can be obtained in the following manner.

Equation (3-15) reduces to

$$\dot{\phi} = \int (a + b \; |V|)^{-1} \frac{\partial h}{\partial x_i} \; d\left(\frac{\partial h}{\partial x_i}\right) \qquad (3\text{-}15a)$$

Now from equations (2-42) and (2-43)

$$(a + b \; |V|)^{-1} = \frac{-\frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}}}{\left|\frac{\partial h}{\partial l}\right|} \qquad (3\text{-}16)$$

Also

$$\left|\frac{\partial h}{\partial l}\right|^2 = \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i}$$

which when differentiated gives

$$\left|\frac{\partial h}{\partial l}\right| d\left|\frac{\partial h}{\partial l}\right| = \frac{\partial h}{\partial x_i} d\left(\frac{\partial h}{\partial x_i}\right) \qquad (3\text{-}17)$$

Substituting equations (3-16) and (3-17) into equation (3-15) gives

$$\dot{\phi} = \int \left(-\frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}} \right) d\left(\left|\frac{\partial h}{\partial l}\right|\right)$$

On integrating the following expression for $\dot{\phi}$ is obtained

$$\dot{\phi} = -\frac{a}{2b}\left|\frac{\partial h}{\partial l}\right| + \frac{2b}{3}\left\{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}\right\}^{3/2} + \text{constant} \qquad (3\text{-}18)$$

3.3.3 Proposed Energy Theorem and Its Application

(i) General Introduction

Energy theorems provide an extremely powerful tool for the theoretical analysis of many physical problems. Via the energy approach, generalised field equations describing the physical phenomenon may be developed.

The energy concept for groundwater flow complying with Darcy's law was first introduced by Muskat (1937) who postulated that the actual distribution of hydraulic heads and flow velocities in a porous medium carrying a fluid under viscous-flow conditions are such as to render the total loss of macroscopic energy of the fluid to a minimum, subject to the existing boundary conditions of the flow system.

Engelund (1953) further extended the concept to two dimensional steady non-Darcy flow. He showed that the integral expression of the rate of dissipation of hydraulic energy in the aquifer region is proportional to the functional of the field equations which he developed for non-Darcy through homogeneous and isotropic media.

In the development presented herein, the author attempts to establish in a rigorous manner the energy theorem for general three-dimensional, transient, two regime flow through aquifers. It is shown that the field equations governing the flow may be obtained by means of this theorem.

(ii) Development of Energy Theorem

Theorem 1: The movement of groundwater through saturated porous aquifer media takes place in such a way that the total rate of dissipation of energy is rendered to a minimum, subject to the existing initial and boundary conditions of the flow system.

Proof:

Consider an arbitrary closed region $\bar{R}$ in the aquifer medium. Let R be the interior volume of $\bar{R}$, and dR be a differential volume of R.

The total rate of energy dissipated in the flow consists of two portions, the first of which is given by

$$\dot{\chi}_1 = \gamma \int_R \dot{\phi} \, dR \qquad (3\text{-}19)$$

The second portion is due to volume compressibility of the elastic aquifer medium. It may be interpreted as the rate of dissipation of elastic energy, and may be expressed as

$$\dot{\chi}_2 = \gamma \int_R (S_s h) \frac{\partial h}{\partial t} \, dR \qquad (3\text{-}20)$$

where S_s is the specific storage of the aquifer medium.

The quantity, $(S_s h)$, may be defined as the volume of water released from storage in a unit volume of an aquifer under a hydraulic head, h.

Let the region R be subdivided into subregions R^N and R^D, which are the non-Darcy and Darcy subregions respectively.

Hence the total rate of energy dissipation is given by

$$\dot{\chi} = \gamma \Big[\int_{R^N} \dot{\phi} \, dR + \int_{R^N} S_s \frac{\partial h}{\partial t} \, h dR$$

$$+ \int_{R^D} \dot{\phi} \, dR + \int_{R^D} S_s \frac{\partial h}{\partial t} \, h \, dR \Big] \qquad (3\text{-}21)$$

where the expressions of $\dot{\phi}$ are given by equations (3-13) and (3-14) respectively.

Substituting equations (3-13) and (3-14) into equation (3-24) gives

$$\dot{\chi}_{/\gamma} = \int_{R^N} \left[\frac{-a}{2b} \left|\frac{\partial h}{\partial l}\right| + \frac{2}{3} b \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b} \right\}^{3/2} + S_s \frac{\partial h}{\partial t} h \right] dR$$

$$+ \int_{R^D} \left[\tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + S_s \frac{\partial h}{\partial t} h \right] dR \qquad (3\text{-}22)$$

The functional, $\dot{\Omega}$, is now defined in accordance with

$$\dot{\Omega} = \dot{\chi}_{/\gamma} \qquad (3\text{-}23)$$

The extremisation of $\dot{\Omega}$ may be established by showing that the vanishing of its first variation leads to an admissible function satisfying the previously derived field equations. The condition of minimisation is then guaranteed by showing that the second variation is a positive definite quantity.

In order to find the first variation, let h (x_i, t) be an admissible function which, together with its second space and first time derivatives, are continuous everywhere in region R. The function, h (x_i, t), must satisfy the initial and prescribed boundary conditions in order to be admissible.

The one-parameter family of "comparison functions" is now defined as

$$H(x_i, t) = h(x_i, t) + \lambda\,(\delta h) \qquad (3\text{-}24)$$

where $\delta h(x_i)$ is an arbitrary function of space that vanishes on the portion B_2 of the flow boundary where h is prescribed, and λ is the real parameter of the family.

The first variation of $\dot{\Omega}$ is given by

$$\delta\dot{\Omega} = \frac{d}{d\lambda} \left[\quad (h + \lambda \delta h) \right]_{\lambda = 0} \qquad (3\text{-}25)$$

Since the function, δ h, is chosen such that $\frac{\partial}{\partial t}$ (δ h) vanishes identically, $\delta\Omega$ may be written as

$$\delta\dot{\Omega} = \int_{R^D} \left[K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j}(\delta h) + S_s \frac{\partial h}{\partial t} \delta h \right] dR$$

$$+ \int_{R^N} \Big[-\frac{a}{2b} \frac{\partial h}{\partial x_i} \frac{\frac{\partial(\delta h)}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} + \frac{2}{3}\cdot\frac{3}{2}\sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}}\ \frac{b}{b} \frac{\partial h}{\partial x_i} \frac{\partial(\delta h)}{\partial x_i}$$

$$+ S_s \frac{\partial h}{\partial t} \delta h \Big] \, dR \qquad (3\text{-}26)$$

The theorem of integration by parts is now employed. It follows that

$$\int_{R^D} K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j}(\delta h) = -\int_{R^D} \frac{\partial}{\partial x_j}\left(K_{ij} \frac{\partial h}{\partial x_i}\right) \delta h \, dR$$

$$+ \int_{B^D} K_{ij} \frac{\partial h}{\partial x_i} n_j \, \delta h \, dR \qquad (3\text{-}27)$$

where B^D is the boundary of region, R^D.

Since δh is chosen to vanish on the boundary, equation (3-27) becomes

$$\int_{R^D} K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j}(\delta h) = -\int_{R^D} \frac{\partial}{\partial x_j}\left(K_{ij} \frac{\partial h}{\partial x_i}\right) \delta h \, dR \qquad (3\text{-}28)$$

Also in a similar manner, it follows that

$$\int_{R^N} \left[-\frac{a}{2b} \frac{\partial h}{\partial x_i} \frac{\frac{\partial}{\partial x_i}(\delta h)}{\left|\frac{\partial h}{\partial l}\right|} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}}\ \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_i}(\delta h) \right] dR$$

$$= \int_{R^N} \delta h \frac{\partial}{\partial x_i} \left\{ \left(-\frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}} \right) \left(-\frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} \right) \right\} dR \qquad (3\text{-}29)$$

Substituting equations (3-28) and (3-29) into equation (3-26) gives

$$\delta\dot{\Omega} = \int_{R^D} \left[-\frac{\partial}{\partial x_j}\left(K_{ij} \frac{\partial h}{\partial x_i}\right) + S_s \frac{\partial h}{\partial t} \right] \delta h \, dR + \int_{R^N} \Bigg[\frac{\partial}{\partial x_i} \Bigg\{ \Bigg(-\frac{a}{2b}$$

$$+ \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}} \Bigg) \left(-\frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} \right) \Bigg\} + S_s \frac{\partial h}{\partial t} \Bigg] \delta h \, dR \qquad (3\text{-}30)$$

Now the two integrals must vanish independently, when $\delta\dot{\Omega}$ is set to zero. Hence it follows that

$$\int_{R^D} \left[- \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial h}{\partial x_i} \right) + S_s \frac{\partial h}{\partial t} \right] \delta h \; dR = 0$$

and

$$\int_{R^N} \left[\frac{\partial}{\partial x_i} \left\{ \left(- \frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}} \right) \left(- \frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} \right) \right\} + S_s \frac{\partial h}{\partial t} \right] \delta h \, dR = 0$$

Since the choice of δh has been made arbitrarily, the integrands of the above two integrals vanish identically. Hence

$$- \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial h}{\partial x_i} \right) + S_s \frac{\partial h}{\partial t} = 0 \qquad (3\text{-}31)$$

$$\text{for } (x_i) \in R^D$$

and

$$\frac{\partial}{\partial x_i} \left[\left(- \frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}} \right) \left(- \frac{\frac{\partial h}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} \right) \right] + S_s \frac{\partial h}{\partial t} = 0 \qquad (3\text{-}32)$$

$$\text{for } (x_i) \in R^N$$

Equations (3-31) and (3-32) are identical to the previously derived field equations of Darcy and non-Darcy flow respectively.

Thus it is now established that the flow of groundwater through porous aquifers takes place in such a way that the total rate of energy dissipation is extremised.

To ensure that the extremised functional corresponds to a minimum, it is sufficient to show that its second variation is a positive definite quantity. The remaining part of the proof is not presented. However, it is pointed out that the functional, $\dot{\Omega}(h)$, is a positive definite quantity as the function, h, and its derivatives appear as squares and products.

(iii) Application of the Energy Theorem

Previously, before the introduction of the energy concept and theorem 1, the functional $[\Omega(h)]_R$ was constructed by applying the Euler-Lagrange equation to the field equations. It has just been established that via energy approach a new functional $\dot{\Omega}(h)$ may be constructed without having to resort to the previously derived field equations, and that the minimisation of $\dot{\Omega}(h)$ leads to the same field equations. Furthermore, if function $h(x_i, t)$ and its time derivative are assumed to be known at time t, and if the time derivative is assumed to remain unvaried between t and $t + \Delta t$, the two functionals may be related by

$$[\Omega(h)]_R = \int_t^{t+\Delta t} \dot{\Omega}(h)\, dt \qquad (3\text{-}33)$$

Thus a physical meaning can be assigned to $[\Omega(h)]_R$. It may be interpreted as the total hydraulic energy dissipated in the interior of the flow region between time t and $t + \Delta t$. Also the terms that have to be added to functional $[\Omega(h)]_R$ to account for additional boundary conditions of the flow system may be interpreted as energy exchanges between the system and its surroundings, which take place across the flow boundary.

It is finally pointed out that the energy theorem just proved remains invariant with respect to the choice of coordinate systems as the rate of energy dissipation is a scalar quantity, which remains unchanged with the change of coordinate systems.

4. Finite Element Analysis of Axisymmetric Well Flow

4.1 Introduction

4.1.1 General Description of the Finite Element Method

In the previous section, variational forms of the field equations were derived and an equivalent variational problem was stated. The problem consists of finding an admissible function which renders a certain functional stationary, subject to the existing initial and boundary conditions of the flow system. The extremised functional was later proved to be a minimum.

An approximate solution of the above variational problem can be obtained by a numerical technique known as the "finite element" method. In this technique, the continuous region of the flow system is subdivided into a finite number of closed subregions termed "finite elements". The finite elements are assumed to be interconnected at a discrete number of points situated on their boundaries. Associated with each element is a chosen function that defines uniquely the hydraulic head distribution within the element in terms of its nodal parameters. The functional over the entire region of flow is assumed to be contributed by each element, and the process of minimisation is accomplished by evaluating the elemental contribution, adding all such contributions, differentiating the resulting functional with respect to the nodal parameter and equating the differential to zero.

The finite element analysis of axi-symmetric flow toward a single well is developed in this section. The analysis considers two flow regimes, namely the non-Darcy regime in the near well zone and the Darcy regime in the remaining outer portion of the aquifer. Anisotropy in aquifers is taken into account only in the Darcy zone. The analysis of non-Darcy flow behaviour in anisotropic aquifers involves complex non-linear velocity-gradient relations and field equations, the theoretical basis and experimental verification of which have not been established. Additional field and laboratory research is still required in order to develop a better understanding of the anisotropic character of the two coefficients of hydraulic resistance in the Forchheimer constitutive relation, namely coefficients 'a' and 'b'.

4.1.2 Basic Definitions and Notation

(i) Definitions

The following definitions have been employed in the formulation of the finite element method presented herein.

Definition (I)

The finite element method is a piecewise approximation process consisting of subdivision of a continuous region into finite elements, evaluation of elemental properties, assemblage of all elements, and solution of a system of algebraic equations.

Definition (II)

A finite element is a closed subregion with the following properties:-

(i) Any admissible element when compatibilised with an adjacent element must have portions of their boundaries mapped into a common inter-element boundary.

(ii) The two elements are considered interconnected at a discrete number of nodal points situated on the common boundary

(iii) Associated with each element is a piecewise function which is expressible as a linear combination of a finite number of independent shape functions; the number of terms in the combination being equal to the number of nodes on the element boundary and the coefficient of the term being the nodal value of the function.

(v) The elements may be classified into:-

"interior elements", which are elements having their closed element boundaries contained within the interior of the entire flow region, and

"exterior elements", which are elements having portions of their boundaries as parts of the entire flow boundary.

(ii) Notation

A subscript notation is employed in the formulation. Both capital letter and small letter subscripts are used. The capital letter subscript refers to a particular node belonging to either an element or the entire flow region. The range of the subscript is from one to the number of nodes on the element boundary, if reference is made to the element, or from one to the total number of nodes, if reference is made to the entire flow region. The small letter subscript, as previously indicated, refers to a particular component along the coordinate axis, and ranges from one to three.

Unless it is indicated, repeated subscripts will be interpreted as summation over the full range and the same subscript will not appear more than twice in the same term of the equation.

4.2 Analysis of Flow in Confined Aquifers

4.2.1 Formulation of Element Matrices

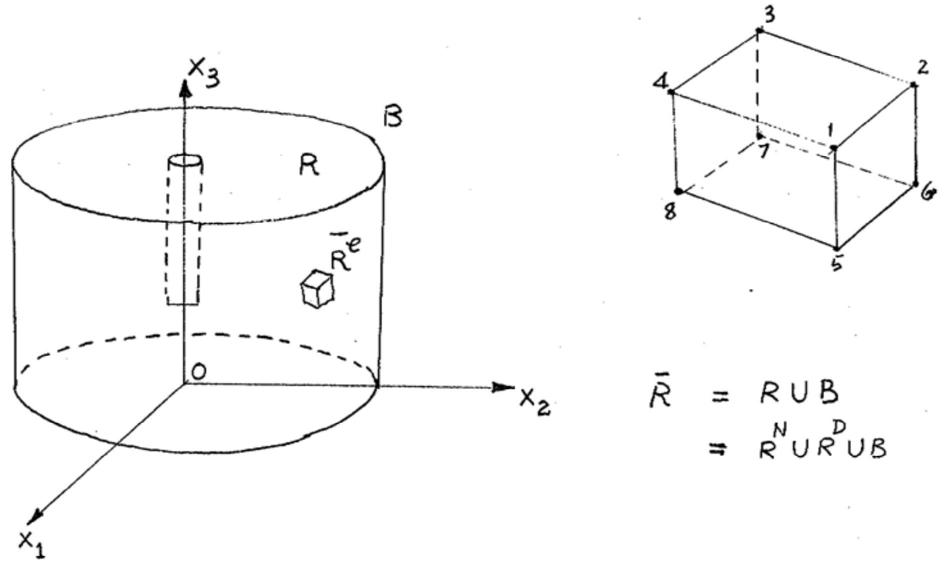

Fig. (4-1) Typical region of a confined aquifer and a finite element.

Consider a general problem of three-dimensional transient flow towards a well penetrating a confined aquifer. A typical flow region $\bar{R}$ is shown in Fig. (4-1). As indicated, $\bar{R}$ is the union of R^N, R^D and B, which are the non-Darcy subregion, the Darcy subregion and the flow boundary respectively.

The functional over $\bar{R}$ may be expressed as the sum of the functionals over R^N, R^D and B. Hence it follows that

$$[\Omega(h)]_{\bar{R}} = [\Omega(h)]_{R^N} + [\Omega(h)]_{R^D} + [\Omega(h)]_B \qquad (4\text{-}1)$$

The expressions for $[\Omega(h)]_{R^N}$ and $[\Omega(h)]_{R^D}$ were derived in Section 2. They may be rewritten as

$$[\Omega(h)]_{R^N} = \int_t^{t+\Delta t}\int_{R^N}\left[-\frac{a}{2b}\left|\frac{\partial h}{\partial l}\right| + \frac{2}{3}b\left\{\left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b}\right\}^{3/2} + S_s h\frac{\partial h}{\partial t}\right] dR\, dt \qquad (4\text{-}2)$$

and

$$[\Omega (h)]_{R^D} = \int_t^{t+\Delta t} \int_{R^D} [\tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + S_s h \frac{\partial h}{\partial t}] dRdt \qquad (4-3)$$

Let the closed boundary, B, of the entire flow region be subdivided into B_1, B_2 which are the prescribed flux portion and the prescribed head portion respectively. The functional over the closed boundary $[\Omega (h)]_B$ is expressible as the sum of the functionals over B_1 and B_2. It follows that

$$[\Omega (h)]_B = \int_t^{t+\Delta t} \int_{B_1} h\bar{q} \, dBdt - \int_t^{t+\Delta t} \int_{B_2} (h - \bar{h}) v_i n_i dBdt \qquad (4-4)$$

In solving the flow problem by the finite element method, the flow region $\bar{R}$ is subdivided into a network consisting of m interconnected finite elements.

Let the closed subregion of a typical element be denoted by $\bar{R}^e$, and the number of nodes situated on the element boundary be n^e.

By employing definition (II), the head distribution within each element is given by

$$h(x_i, t) = N_I (x_i) h_I (t) \qquad (4-5)$$

where $N_I(x_i)$ are piecewisely defined functions of coordinates (x_1, x_2, x_3) within the element, $h_I(t)$ are the nodal values at time t of function h, and the repeated subscripts, I, represent summation over the full range from 1 to n^e.

The functional over the entire flow region, $[\Omega (h)]_{\bar{R}}$, is also expressible as the sum of the functionals over the finite elements, $\Omega^e (h)$. Hence

$$[\Omega (h)]_{\bar{R}} = \sum_{e=1}^{m} \Omega^e (h) \qquad (4-6)$$

The evaluation of elemental contributions is accomplished by evaluating firstly the contribution from the interior element and secondly the contribution from the exterior element. In the process of evaluation, it is assumed that the element is sufficiently small so that $\bar{R}^e$ may be considered to belong to either R^N or R^D. The criterion for determining whether $\bar{R}^e$ belongs to R^N or R^D is as follows:

If the Reynolds number at the centroid of the element is greater than the critical Reynolds number, the element will be considered to belong to R^N otherwise it belongs to R^D.

(i) Interior Elements.

(a) Elements belonging to R^N.

For interior elements belonging to R^N, the functional over $\bar{R}^e$ is given by

$$\Omega^e(h) = \int_t^{t+\Delta t} \int_{R^e} \left[-\frac{a}{2b} \left|\frac{\partial h}{\partial 1}\right| + \frac{2}{3} b \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial 1}\right|}{b} \right\}^{3/2} + S_s \, h \, \frac{\partial h}{\partial t} \right] dRdt \qquad (4\text{-}7)$$

Differentiating equation (4-7) with respect to h_I gives

$$\frac{\partial \Omega^e(h)}{\partial h_I} = \int_t^{t+\Delta t} \int_{R^e} \left[-\frac{a}{2b} \frac{\partial}{\partial h_I} \left(\left|\frac{\partial h}{\partial 1}\right| \right) + b \left\{ \left(\frac{a}{2b} \right)^2 + \frac{\left|\frac{\partial h}{\partial 1}\right|}{b} \right\}^{\frac{1}{2}} \frac{1}{b} \frac{\partial}{\partial h_I} \left|\frac{\partial h}{\partial 1}\right| + S_s \, h \, \frac{\partial}{\partial h_I} \left(\frac{\partial h}{\partial t} \right) + S_s \frac{\partial h}{\partial t} \frac{\partial h}{\partial h_I} \right] dRdt \qquad (4\text{-}8)$$

where it should be noted that subscript s is not a small letter subscript. S_S merely denotes specific storage of the aquifer.

From equation (4-5)

$$\frac{\partial h}{\partial x_i} = \frac{\partial N_I}{\partial x_i} h_I \qquad (4\text{-}9)$$

Contracting subscript i gives

$$\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} = \frac{\partial N_I}{\partial x_i} h_I \frac{\partial N_J}{\partial x_i} h_J \qquad (4\text{-}10)$$

Now

$$\left|\frac{\partial h}{\partial 1}\right|^2 = \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} \qquad (4\text{-}11)$$

Differentiating equation (4-11) with respect to h_I gives

$$\frac{\partial}{\partial h_I} \left|\frac{\partial h}{\partial l}\right| = \frac{\frac{\partial h}{\partial x_i} \frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial x_i}\right)}{\left|\frac{\partial h}{\partial l}\right|} \tag{4-12}$$

Also from equation (4-9)

$$\frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial x_i}\right) = \frac{\partial N_I}{\partial x_i} \tag{4-13}$$

Substituting equations (4-9) and (4-13) into equation(4-12) results in

$$\frac{\partial}{\partial h_I} \left|\frac{\partial h}{\partial l}\right| = \frac{\frac{\partial N_J}{\partial x_i} h_J \frac{\partial N_I}{\partial x_i}}{\left|\frac{\partial h}{\partial l}\right|} \tag{4-14}$$

Since N_I are functions which do not vary with time, it follows that

$$h \frac{\partial}{\partial h_I}\left(\frac{\partial h}{\partial t}\right) = N_J h_J \frac{\partial N_I}{\partial t} = 0 \tag{4-15}$$

and

$$\frac{\partial h}{\partial t} \frac{\partial h}{\partial h_I} = \frac{\partial h_J}{\partial t} N_J N_I \tag{4-16}$$

Substituting equations (4-14) and (4-16) into equation (4-8) gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_t^{t+\Delta t} \int_{R^e} \left[-\frac{a}{2b} + \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b} \right\}^{\frac{1}{2}} \right] \frac{1}{\left|\frac{\partial h}{\partial l}\right|} \frac{\partial N_J}{\partial x_i} \frac{\partial N_I}{\partial x_i} h_J \, dRdt + \int_t^{t+\Delta t} \int_{R^e} S_s \frac{\partial h_J}{\partial t} N_J N_I \, dRdt \tag{4-17}$$

Introducing the following equations

$$A = \left[-\frac{a}{2b} + \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b} \right\}^{\frac{1}{2}} \right] \frac{1}{\left|\frac{\partial h}{\partial l}\right|} \tag{4-18a}$$

$$C^{e}_{JI} = \int_{R^e} A \frac{\partial N_J}{\partial x_i} \frac{\partial N_I}{\partial x_i} \, dR \qquad (4\text{-}18b)$$

$$D^{e}_{JI} = \int_{R^e} S_s \; N_J \, N_I \, dR \qquad (4\text{-}18c)$$

Substituting these equations into equation (4-17) gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_t^{t+\Delta t} C^e_{JI} h_J \, dt + \int_t^{t+\Delta t} D^e_{JI} \frac{\partial h_J}{\partial t} \, dt \qquad (4\text{-}19)$$

where J and I range from 1 to the number of nodes on the element boundary, n^e.

(b) Elements Belonging to R^D

For elements belonging to R^D, the functional over $\bar{R}^e$ is given by

$$\Omega^e(h) = \int_t^{t+\Delta t} \int_{R^e} \left[\tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + S_s \; h \frac{\partial h}{\partial t} \right] dR dt \qquad (4\text{-}20)$$

Differentiating equation (4-20) with respect to h_I gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_t^{t+\Delta t} \int_{R^e} \left[K_{ij} \frac{\partial h}{\partial x_j} \frac{\partial}{\partial h_I} \left(\frac{\partial h}{\partial x_i}\right) + S_s \; h \frac{\partial}{\partial h_I} \left(\frac{\partial h}{\partial t}\right) + S_s \frac{\partial h}{\partial t} \frac{\partial h}{\partial h_I} \right] dR dt \qquad (4\text{-}21)$$

Substituting equations (4-13), (4-15) and (4-16) into equation (4-21) gives

$$\frac{\partial \Omega^e}{\partial h_I} \quad \int_t^{t+\Delta t} \int_{R^e} \left[K_{ij} \frac{\partial N_J}{\partial x_j} h_J \frac{\partial N_I}{\partial x_i} + S_s \frac{\partial h_J}{\partial t} N_J N_I \right] dR dt \qquad (4\text{-}22)$$

Introducing $$\acute{C}^{e}_{JI} = \int_{R^e} K_{ij} \frac{\partial N_J}{\partial x_i} \frac{\partial N_I}{\partial x_i} \, dR \qquad (4\text{-}23)$$

Substituting equations (4-18c) and (4-23) into equation (4-22) gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_t^{t+\Delta t} C'^e_{JI} \, h_J dt + \int_t^{t+\Delta t} D^e_{JI} \frac{\partial h_J}{\partial t} \, dt \qquad (4\text{-}24)$$

(ii) Exterior Elements

In evaluating the functional contributed by the exterior element, allowance must be made for the additional boundary conditions on the element boundary. Accordingly, extra terms must be added to the functional already derived for the interior element. These terms only exist on the exterior boundary and vanish elsewhere. The additional boundary conditions of confined flow are prescribed flux and prescribed head conditions. They may be dealt with in the following manner:-

Let B_1^e and B_2^e be the exterior portions of the element boundary where the flux and the head function are prescribed respectively. The additional term existing on B_1^e is given by

$$\int_t^{t+\Delta t} \int_{B_1^e} h\bar{q} \, dBdt$$

and the additional existing on B_2^e is

$$- \int_t^{t+\Delta t} \int_{B_1^e} (h - \bar{h}) \; v_i \, n_i \, dBdt$$

The resulting functional over $\bar{R}^e$ may now be written as

$$[\Omega^e(h)]_{\bar{R}^e} = \int_t^{t+\Delta t} \int_{R^e} \left[-\frac{a}{2b} \left|\frac{\partial h}{\partial 1}\right| + \frac{2}{3b} \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial 1}\right|}{b} \right\}^{3/2} \right.$$

$$\left. + \; S_s \, h \frac{\partial h}{\partial t} \right] dRdt$$

$$+ \int_t^{t+\Delta t} \int_{B_1^e} h\bar{q} \; dBdt$$

$$- \int_{t}^{t+\Delta t} \int_{B_2^e} (h - \bar{h}) \; v_i \, n_i \; dBdt \qquad (4\text{-}25)$$

Since the admissible function h is chosen to automatically satisfy the prescribed head condition on the entire flow boundary, the extra term contributed by boundary portion B_2^e may be dropped from equation (4-25a). Hence equation (4-25a) becomes

$$[\Omega^e(h)]_{\bar{R}^e} = \int_{t}^{t+\Delta t} \int_{R^e} \Big[- \frac{a}{2b} \left|\frac{\partial h}{\partial l}\right| + \frac{2}{3} b \left\{ \left(\frac{a}{2b}\right)^2 + \frac{\left|\frac{\partial h}{\partial l}\right|}{b} \right\}^{3/2}$$

$$+ S_s \, h \, \frac{\partial h}{\partial t} \Big] dRdt + \int_{t}^{t+\Delta t} \int_{B_1^e} h\bar{q} \; dBdt \qquad (4\text{-}25b)$$

Thus differentiating equation (4-25b) with respect to h_I gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_{t}^{t+\Delta t} C_{JI}^e \; h_J \; dt + \int_{t}^{t+\Delta t} D_{JI}^e \; \frac{\partial h_J}{\partial t} \; dt + \int_{t}^{t+\Delta t} F_I^e \; dt \qquad (4\text{-}26)$$

$$\text{where } F_I^e = \int_{B_1^e} \bar{q} \; N_I \; dB \qquad (4\text{-}27)$$

Similarly for elements belonging to R^D the expression for the functional may be written as

$$[\Omega(h)]_{\bar{R}^e} = \int_{t}^{t+\Delta t} \int_{R^e} \left[\tfrac{1}{2} K_{ij} \frac{\partial h}{\partial x_j} \frac{\partial h}{\partial x_i} + S_s \; h \; \frac{\partial h}{\partial t} \right] dR \; dt$$

$$+ \int_{t}^{t+\Delta t} \int_{B_1^e} h \, \bar{q} \; dB \; dt \qquad (4\text{-}28)$$

Differentiating with respect to h_I gives

$$\frac{\partial \Omega^e}{\partial h_I} = \int_t^{t+\Delta t} \acute{C}^e_{JI} \; h_J dt + \int_t^{t+\Delta t} D^e_{JI} \; \frac{\partial h_J}{\partial t} \; dt + \int_t^{t+\Delta t} F^e_I \; dt \tag{4-29}$$

(iii) Element Matrices

The above formulation leads to various element matrices, which have been expressed in subscript notation. The expressions are given in Equations (4-18b), (4-18c), (4-23) and (4-27). They can be converted into compact matrix forms as follows:-

Let $[C^e]$, $[\acute{C}^e]$, $[D^e]$ and $[F^e]$ denote the element matrices having matrix elements C^e_{IJ}, $\acute{C}^e_{IJ}$, D^e_{IJ} and F^e_I respectively. By employing the matrix notation, the following equations may be written

$$[C^e] = \int_{R^e} A \; [S]^T [S] \; dR \tag{4-30}$$

$$[\acute{C}^e] = \int_{R^e} [S]^T [K][S] \; dR \tag{4-31}$$

$$[D^e] = \int_{R^e} S_s \; [N]^T [N] \; dR \tag{4-32}$$

$$[F^e] = \int_{B_1^e} \bar{q} \; [N]^T \; dB \tag{4-33}$$

where A is a non-linear coefficient given by

$$A = \left[-\frac{a}{2b} + \left\{ \left(\frac{a}{2b}\right)^2 + \left|\frac{\partial h}{\partial l}\right| \right\}^{\frac{1}{2}} \right] \frac{1}{\left|\frac{\partial h}{\partial l}\right|} \tag{4-34a}$$

in which $\left|\frac{\partial h}{\partial l}\right|^2 = \frac{\partial N_I}{\partial x_i} \frac{\partial N_J}{\partial x_i} h_I \; h_J$ (4-34b)

$[S]^T$ is the transpose of matrix $[S]$, which is given by

$$[S]^T = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & - - - - - - - - - & \frac{\partial N_{n^e}}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & - - - - - - - - - & \frac{\partial N_{n^e}}{\partial x_2} \\ \frac{\partial N_1}{\partial x_3} & - - - - - - - - - & \frac{\partial N_{n^e}}{\partial x_3} \end{bmatrix} \qquad (4\text{-}35)$$

[K] is the hydraulic conductivity matrix, which may be written as

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \qquad (4\text{-}36)$$

[N] is the shape function matrix, which may be written as

$$[N] = [N_1 \ldots\ldots \ldots\ldots N_{n^e}] \qquad (4\text{-}37)$$

In a similar manner the matrix equations relating the differentials of functional, Ω^e, and the nodal values of function h may be written. The matrix equation for exterior elements in the non-Darcy subregion is obtained from equation (4-26). It is given by

$$\left[\frac{\partial \Omega^e}{\partial h}\right] = \int_t^{t+\Delta t} [C^e][h^e]\,dt + \int_t^{t+\Delta t} [D^e]\left[\frac{\partial h^e}{\partial t}\right] dt + \int_t^{t+\Delta t} [F^e]\,dt \qquad (4\text{-}38)$$

where

$$\left[\frac{\partial \Omega}{\partial h}^e\right] = \begin{bmatrix} \frac{\partial \Omega^e}{\partial h_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \Omega^e}{\partial h_{n^e}} \end{bmatrix} \qquad [h^e] = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h_{n^e} \end{bmatrix}$$

The matrix equation for an interior element in the non-Darcy sub-region is obtained by dropping the last integral term on the right hand side of equation (4-38). Hence it follows that

$$\left[\frac{\partial\Omega}{\partial h}\right]^e = \int_t^{t+\Delta t} [C^e][h^e]\,dt + \int_t^{t+\Delta t} [D^e]\left[\frac{\partial h}{\partial t}\right]^e dt \qquad (4\text{-}39)$$

The matrix equations for exterior and interior elements in the Darcy sub-region are obtained from equations (4-38) and (4-39) respectively by merely replacing matrix (C^e) in these equations by matrix $(\acute{C}^e)$.

4.2.2 Element Matrices for Triangular Ring Elements

The formulation of the element matrix just presented is a generalised procedure that is applicable for three-dimensional well flow. Many problems of flow toward wells encountered in practice are usually axi-symmetric flow problems. For the problems of this kind, the formulation of the element matrix may be simplified by employing the two-dimensional cylindrical coordinate system, namely (r, z). The entire flow region may be subdivided into a finite number of ring elements concentric about the vertical axis of the well. These elements are readily generated by revolving plane sections about the vertical z-axis. A typical triangular ring element is shown in Fig. (4-2).

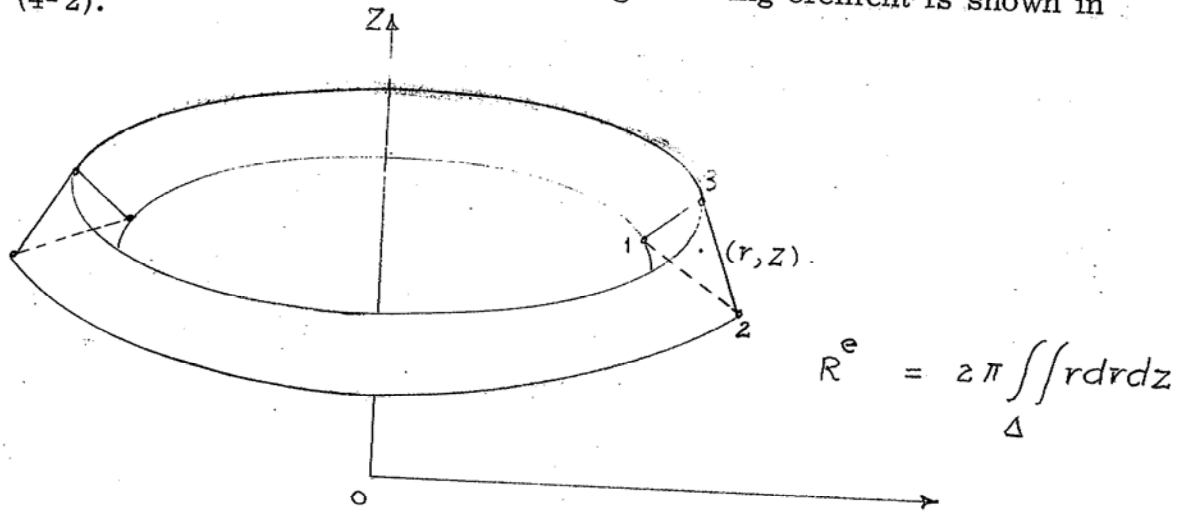

Fig. (4-2): A typical triangular ring element.

If the hydraulic head distribution at point (r, z) in the element is represented by a linear function of r and z in terms of nodal values, then it can be shown (Zienkiewicz and Cheung, 1967) that the shape functions may be expressed as:-

$$N_I = a_I + b_I r + c_I z \qquad (4\text{-}40)$$

where a_1, b_1, c_1 are given by

$$a_1 = (r_2 z_3 - r_3 z_2)/2\Delta \qquad (4\text{-}41a)$$

$$b_1 = \frac{z_2 - z_3}{2\Delta} \qquad (4\text{-}41b)$$

$$c_1 = \frac{r_3 - r_2}{2\Delta} \qquad (4\text{-}41c)$$

The remaining coefficients are obtained by cyclic permutation of subscripts, and Δ is the area of triangle 1-2-3, which is given by

$$\Delta = \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} \qquad (4\text{-}41d)$$

Now from equation (4-40), it follows that

$$\frac{\partial N_I}{\partial r} = b_I$$

$$\frac{\partial N_I}{\partial z} = c_I$$

Hence matrix $[S]^T$ may be written as

$$[S]^T = \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad (4\text{-}42a)$$

In a two dimensional coordinate system the hydraulic conductivity matrix is

$$[K] = \begin{bmatrix} K_{rr} & K_{rz} \\ K_{rz} & K_{zz} \end{bmatrix} \qquad (4\text{-}42b)$$

The shape function matrix for triangular element is

$$[N] = [\, N_1, N_2, N_3 \,] \qquad (4\text{-}42c)$$

in which the expressions for N_1, N_2 and N_3 are given by equation (4-40).

The expressions for element matrices $[C^e]$ and $[\acute{C}^e]$ can be obtained by direct integration and noting that dR^e has to be replaced by

$$dR^e = 2\pi \, r dr dz \tag{4-42d}$$

The array elements of matrices $[C^e]$ are now expressible as

$$C^e_{IJ} = \int_{R^e} \left[A \frac{\partial N_I}{\partial x_i} \frac{\partial N_J}{\partial x_i} \right] dR$$

which may be approximated by:-

$$C^e_{IJ} = \frac{2\pi \bar{r}}{4\triangle} A \left(b_I b_J + c_I c_J \right) \tag{4-43}$$

where $\bar{r}$ is the centroidal radius of the triangular plane section, and A is the non-linear coefficient given by equations (4-34a) and (4-34b).

Also the array elements of $[C^e]$ are given by

$$C^e_{IJ} = \int_{R^e} \left[K_{ij} \frac{\partial N_I}{\partial x_i} \frac{\partial N_J}{\partial x_j} \right] dR$$

Substituting for various terms on the right hand side and expanding, the above equation becomes

$$C^e_{IJ} = \frac{2\pi \bar{r}}{4\triangle} \left(K_{rr} b_I b_J + K_{rz} b_I c_J + K_{zr} c_I b_J + K_{zz} c_I c_J \right) \tag{4-44}$$

in which repeated subscripts r and z do not imply the usual summation convention.

The integration for the matrix elements of matrices $[D^e]$ and $[F^e]$ requires more labour. The two matrices have been evaluated by Parekh(1967), and are rewritten as

$$D^e = \frac{2\pi r S_s}{3} \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \tag{4-45}$$

$$F^e = 2\pi \bar{r} \bar{q} L \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} \tag{4-46}$$

in which it is assumed in equation (4-46) that side 1-2 of triangle 1-2-3 corresponds to the exterior boundary portion where the flux is prescribed, and the length of side 1-2 of the triangle is denoted by L.

4.2.3 Element Matrices for Isoparametric Ring Elements

(i) One-dimension Elements

Problems of one-dimensional flow toward a pumped well fully penetrating an isotropic aquifer is simplified by the use of non-dimensional isoparametric elements.

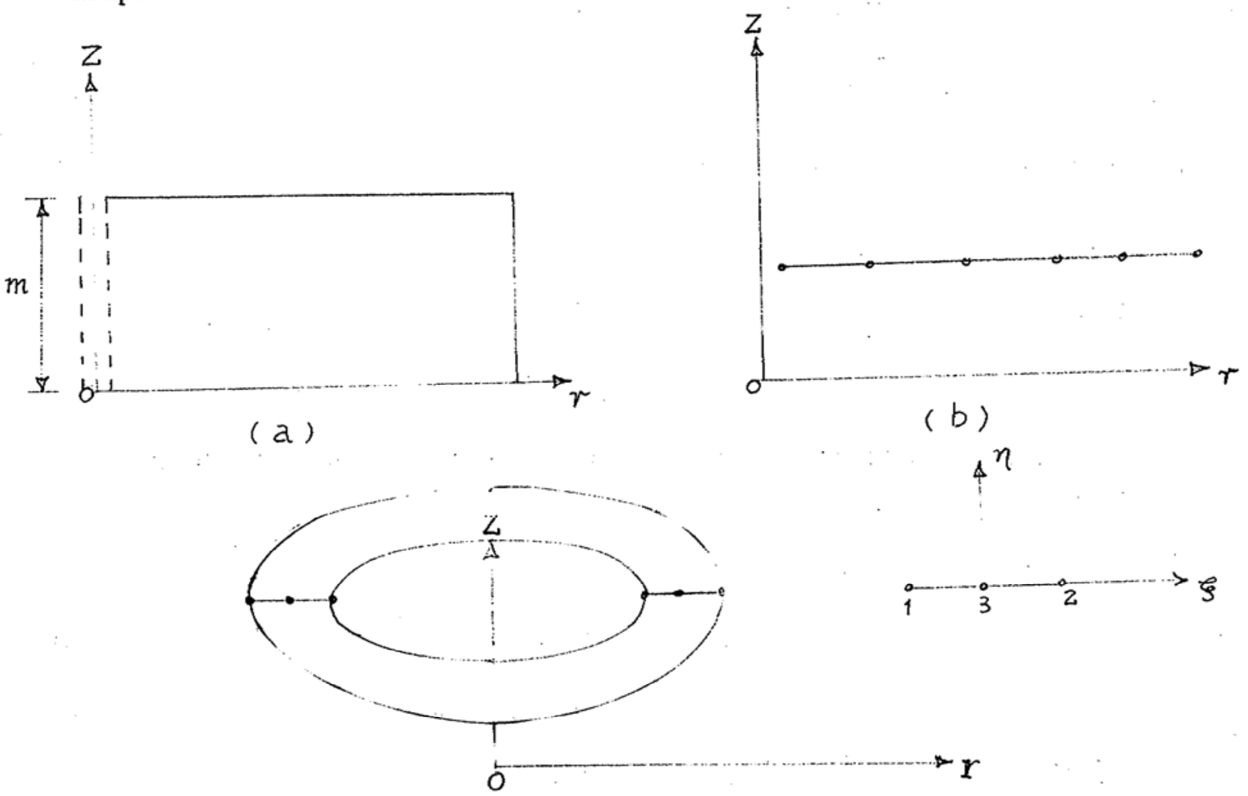

Fig. (4-3): Idealised one-dimensional region and one-dimensional isoparametric elements.

Consider a typical well-aquifer system shown in Fig. (4-3a). Since the velocity of flow is in the radial direction, it is sufficient to find the hydraulic head distribution along any radial line. Accordingly, the two-dimensional region in r-z plane shown in Fig. (4-3a) may be reduced to a radial line shown in Fig. (4-3b). The line is subdivided into a network of line elements. A typical element is shown in Fig. (4-3c). The planar ring section is readily generated by revolving the line element above the z-axis.

Let ξ be a local coordinate associated with each of the line elements. The coordinate is so determined as to give $\xi = 0$ at node 3, $\xi = 1$ at node 2, and $\xi = -1$ at node 1. The relationship between the radial coordinate and ξ- coordinate is given by

$$r = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 \qquad (4\text{-}47)$$

The derivation of the shape functions for a family of isoparametric elements has been presented by Ergatoudis, et al (1968). The shape functions for the above line element are given by

$$[N_1, N_2, N_3] = [-0.5(\xi - \xi^2), -0.5(\xi + \xi^2), (1 - \xi^2)] \qquad (4\text{-}52)$$

The differentials dr and $d\xi$ are related by

$$dr = \frac{dr}{d\xi} \cdot d\xi$$

from which it follows that

$$dr = \left(\frac{dN_I}{d\xi} r_I\right) d\xi = J d\xi \qquad (4\text{-}53)$$

where the repeated subscripts represent summation over the full range and J is the Jacobian transformation matrix.

The expression for J is

$$J = 0.5(-1+2\xi)r_1 + 0.5(1+2\xi) r_2 + (-2\xi) r_3 \qquad (4\text{-}54)$$

Also

$$\frac{dN_I}{dr} = \frac{dN_I}{d\xi} \frac{d\xi}{dr} = \frac{1}{J} \frac{dN_I}{d\xi}$$

Hence the slope matrix, [S] for the 3-node line element becomes

$$[S] = \frac{dN}{dr}$$

$$[S] = \frac{1}{J} [0.5(-1 + 2\xi), 0.5(1 + 2\xi), -2\xi] \qquad (4\text{-}55)$$

Now from equation (4-30) the element matrix $[C^e]$ is given by

$$[C^e] = \int_{R^e} A [S]^T [S] \, dR \qquad (4\text{-}56)$$

where $dR = 2\pi r\,dr$

Equation (4-56) reduces to

$$[C^e] = 2\pi \int_{-1}^{1} A\,[S]^T [S]\; N_I\, r_I\, J d\xi \qquad (4\text{-}57)$$

where the definite integral may be evaluated numerically by employing the gaussian quadrature formulae. Examples of numerical integration by the gaussian formulae have been cited by Zienkiewicz (1967). On applying the 3 point quadrature formula to equation (4-57) and multiplying, the following expression results:-

$$[C^e] = 2\pi \sum_{i=1}^{3} \frac{A(\xi_i)}{J(\xi_i)}\, N_I(\xi_i)\; r_I\, W(\xi_i) \times \begin{bmatrix} 0.25(-1+2\xi_i)^2 & 0.25(4\xi_i^2 - 1) & -0.5(1-2\xi_i)^2 \\ 0.25(4\xi_i^2-1) & 0.25(1+2\xi_i)^2 & 0.5(1-4\xi_i^2) \\ -.5(1-2\xi_i)^2 & .5(1-4\xi_i)^2 & (1-2\xi_i)^2 \end{bmatrix} \qquad (4\text{-}58)$$

where $A(\xi_i)$, $N_I(\xi_i)$, and $J(\xi_i)$ are functions of the ξ-coordinate evaluated at the gaussian points, and $W(\xi_i)$ are the values of the weighting coefficient at the gaussian points.

Similar expressions for matrices $[C^e]$ and $[D^e]$ can be obtained

$$[C^e] = 2\pi \int_{-1}^{1} [K][S]^T [S]\; N_I r_I\, J d\xi \qquad (4\text{-}59)$$

$$[D^e] = 2\pi \int_{-1}^{1} S_s\, [N]^T [N]\; N_I\, r_I\, J d\xi \qquad (4\text{-}60)$$

(ii) Quadrilateral Elements

The accuracy of the finite element solution of the two-dimensional axi-symmetric well flow may be improved by the use of quadrilateral elements. This type of element provides higher forms of approximation to the hydraulic head function. Their use allows an appreciable reduction in the total number of nodes in the flow region for a given degree of accuracy. A typical 4-node element is shown in Fig. (4-4).

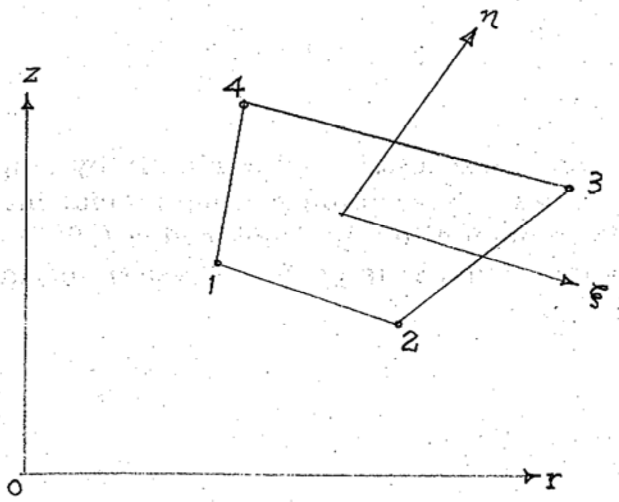

Fig. (4-4): A 4-node quadrilateral element.

A quadrilateral ring can be generated by revolving this element about the z-axis.

Let a system of local coordinates (ξ, η) be associated with each of the elements. These coordinates are so determined as to give $\eta = -1$ on side 1-2, $\eta = +1$ on side 4-3, $\xi = 1$ on side 2-3, and $\xi = -1$ on side 1-4. The relationships between the r-z coordinates and ξ-η coordinates are given by

$$r = N_I(\xi, \eta)\, r_I \qquad (4\text{-}61a)$$

and

$$z = N_I(\xi, \eta)\, z_I \qquad (4\text{-}61b)$$

where I ranges from 1 to 4, and N_I are the shape functions.

The expressions for N_I have been developed by Zienkiewicz (1967). They are written as

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\ ; \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta); \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta) \qquad (4\text{-}62)$$

The differential operators with respect to r and z and that with respect to

ξ and η are related by

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \qquad (4\text{-}63)$$

where J is the Jacobian matrix, which is given by

$$[J] = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \qquad (4\text{-}64)$$

$$J = \frac{1}{4} \begin{bmatrix} -(1-\eta), & (1-\eta), & (1+\eta), & -(1+\eta) \\ -(1-\xi), & -(1+\xi), & (1+\xi), & (1-\xi) \end{bmatrix} \begin{bmatrix} r_1 & z_1 \\ r_2 & z_2 \\ r_3 & z_3 \\ r_4 & z_4 \end{bmatrix}$$

Also $\quad dR = 2\pi r \, dr dz$

and $\quad dr dz = |J| \, d\xi \, d\eta \qquad (4\text{-}65)$

where $|J|$ is the determinant of the Jacobian matrix.

The slope matrix, $[S]$, is expressible as

$$[S] = [J]^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial \xi}, & \frac{\partial N_2}{\partial \xi}, & \frac{\partial N_3}{\partial \xi}, & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta}, & \frac{\partial N_2}{\partial \eta}, & \frac{\partial N_3}{\partial \eta}, & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \qquad (4\text{-}66)$$

The element matrices $[C^e]$, $[C'^e]$, $[D^e]$ are given by

$$C^e = 2\pi \int_{-1}^{1} \int_{-1}^{1} A \, [S]^T [S] \, N_I \, r_I \, |J| \, d\xi \, d\eta \qquad (4\text{-}67a)$$

$$[C^e] = 2\pi \int_{-1}^{1} \int_{-1}^{1} [S]^T [K] [S] \; N_I \, r_I \; |J| \; d\xi \; d\eta \qquad (4\text{-}67b)$$

$$[D^e] = 2\pi \int_{-1}^{1} \int_{-1}^{1} S_s \; [N]^T [N] \; N_I \; r_I \; |J| \; d\xi \; d\eta \qquad (4\text{-}67c)$$

where the numerical integration is accomplished by using the gaussian quadrature formula.

4.2.4 Assemblage of Elements

In the assembling process, all elements are assembled through the specification of the reduced compatability condition, which requires that the nodal values of the function be the same at coincident nodes of adjacent elements and also equal to the prescribed value on the boundary portion where the function is prescribed.

Thus on assembling, the functional for the entire flow region becomes

$$[\Omega(h_I)]_{\bar{R}} = \sum_e \Omega^e(h_I) \qquad (4\text{-}68)$$

for I = 1 ....n

where the summation is taken over the elements adjacent to the I-th nodal point, and subscript I ranges from one to the total number of nodes in the entire flow region.

The minimisation of $[\Omega(h_I)]_{\bar{R}}$ requires that

$$\frac{\partial[\Omega(h_I)]_{\bar{R}}}{\partial h_I} = \sum_e \frac{\partial \Omega^e}{\partial h_I} = 0 \qquad (4\text{-}69)$$

for I = 1 ... .n

where n is the total number of nodes in the flow region.

The expressions for $\frac{\partial \Omega^e}{\partial h_I}$ have been obtained for both elements in the non-Darcy subregion and elements in the Darcy subregion. The general expression is now rewritten as

$$\frac{\partial \Omega^{e}}{\partial h_I} = \int_t^{t+\Delta t} C^e_{JI}\, h_J\, dt + \int_t^{t+\Delta t} D^e_{JI}\, \frac{\partial h_J}{\partial t}\, dt + \int_t^{t+\Delta t} F^e_I\, dt \qquad (4\text{-}70)$$

where for an element in the Darcy subregion, C^e_{JI} is replaced by C'^e_{JI}

Substituting the above equation into equation (4-69) gives

$$\sum_e \int_t^{t+\Delta t} C^e_{JI}\, h_J\, dt + \sum_e \int_t^{t+\Delta t} D^e_{JI}\, \frac{h_J}{t}\, dt + \sum_e \int_t^{t+\Delta t} F^e_I\, dt = 0 \qquad (4\text{-}71)$$

The following gross matrices are now introduced.

$$C_{JI} = \sum_{e=1}^{m} C^e_{JI} \qquad (4\text{-}71a)$$

$$D_{JI} = \sum_{e=1}^{m} D^e_{JI} \qquad (4\text{-}71b)$$

$$F_I = \sum_{e=1}^{m} F^e_I \qquad (4\text{-}71c)$$

Thus equation (4-71) may be rewritten as

$$\int_t^{t+\Delta t} C_{JI}\, h_J\, dt + \int_t^{t+\Delta t} D_{JI}\, \frac{\partial h_J}{\partial t}\, dt + \int_t^{t+\Delta t} F_I\, dt = 0 \qquad (4\text{-}72)$$

where J and I now range from one to the total number of nodes in the entire flow region.

Equation (4-72) is expressed in the integral form. In order to carry out the time integration, it is assumed that all nodal values of h_J and F_I are known at earlier time t, and that the nodal values vary linearly over time increment Δt which is made sufficiently small.

Hence equation (4-72) may now be integrated to result in

$$\int_{t}^{t+\Delta t} C_{JI} \; h_J \, dt + D_{JI} \, (h_J^{t+\Delta t} - h_J^{t}) + (F_I^{t+\Delta t} + F_I^{t}) \, \frac{\Delta t}{2} = 0 \quad (4\text{-}73)$$

where the superscripts denote the time at which the nodal values are evaluated.

The remaining integral term in equation (4-73) involves both C_{JI} and h_J, which vary with time. This is because C_{JI} is associated with the non-linear coefficient, A, which has been given in equations (4-34a) and (4-34b) in terms of the nodal values of the function. To avoid further complication, the above integral is approximated by

$$\int_{t}^{t+\Delta t} C_{JI} \; h_J dt = \frac{(C_{JI}^{t+\Delta t} + C_{JI}^{t})}{2} \; \frac{(h_J^{t+\Delta t} + h_J^{t})}{2} \; \Delta t \quad (4\text{-}74)$$

Substituting this into equation (4-73) gives

$$(C_{JI}^{t+\Delta t} + C_{JI}^{t}) \, (h_J^{t+\Delta t} + h_J^{t}) \, \frac{\Delta t}{4} + D_{JI} \, (h_J^{t+\Delta t} - h_J^{t}) + (F_I^{t+\Delta t} + F_I^{t}) \, \frac{t}{2} = 0 \quad (4\text{-}75)$$

which may be rearranged to give

$$\tfrac{1}{2}(h_J^{t} + h_J^{t+\Delta t}) \left[\frac{\Delta t}{2} \, (C_{JI}^{t} + C_{JI}^{t+\Delta t}) + D_{JI} \right] = D_{JI} h_J^{t} - \frac{\Delta t}{4} (F_I^{t} + F_I^{t+\Delta t}) \quad (4\text{-}76)$$

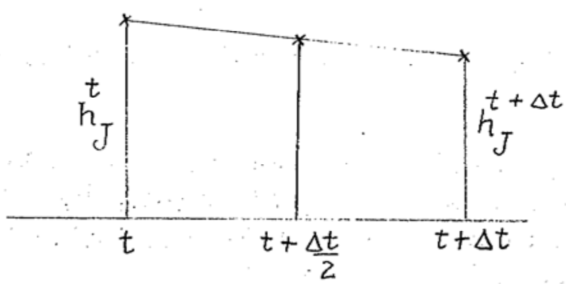

Fig (4-5): Nodal values and their variation over Δt.

Now let $t + \frac{\Delta t}{2}$ denote the mid-time of t and t +Δt. It follows that

$$h_J^{t+\Delta t/2} = \frac{1}{2} (h_J^t + h_J^{t+\Delta t/2}) \quad (4\text{-}77a)$$

$$F_I^{t+\Delta t/2} = \frac{1}{2} (F_I^{t+\Delta t} + F_I^t) \quad (4\text{-}77b)$$

and

$$C_{JI}^{t+\Delta t/2} = (C_{JI}^t + C_{JI}^{t+\Delta t})/2 \quad (4\text{-}77c)$$

On substituting these into equation (4 - 76), the following equation is obtained

$$\left(\frac{\Delta t}{2} C_{JI}^{t+\frac{\Delta t}{2}} + D_{JI} \right) h_J^{t+\frac{\Delta t}{2}} = D_{JI} h_J^t - \frac{\Delta t}{2} F_I^{t+\frac{\Delta t}{2}} \quad (4\text{-}78)$$

Also from equation (4-77a), it follows that

$$h_J^{t+\Delta t} = 2h_J^{t+\frac{\Delta t}{2}} - h_J^t \quad (4\text{-}79)$$

The last two equations provide a compact scheme for final time integration. The forward time integration starts at the initial time, t = o, at which the initial condition is known. At the beginning of the first time step, the nodal head values, h_J^o, are specified by the initial condition. These values may be substituted for h_J^t in equation (4-78) and used in solving for $h_J^{t+\Delta t/2}$, which are then substituted into equation (4-79) to result in the head values at the end of the time step. The currently obtained head values thus correspond to h_J^t at the beginning of the second time step, and may be used in obtaining the head values at the end of this new time step. The procedure is repeated until all the nodal head values prior to a specified time have been determined.

Equation (4-78) is a set of linear and mildly non-linear algebraic equations. The non-linear equations involve the non-linear coefficients, $C_{JI}^{t+\Delta t/2}$, which are contributed by elements in the non-Darcy subregion. These coefficients have to be evaluated in terms of the unknown nodal values of h at time t +Δt/2, as they are associated with coefficient A of the finite elements. However, provided that the values of h_J^t and $F_I^{t+\Delta t/2}$ at all the nodal points of the flow region are known, it is possible to solve for $h_J^{t+\frac{\Delta t}{2}}$ iteratively. The procedure is to firstly calculate matrix $[C^{t+\frac{\Delta t}{2}}]$ in terms of h_J^t, which are used as starting values, and solve for $h_J^{t+\Delta t/2}$, then use the values of $h_J^{t+\Delta t/2}$ just obtained to reform the non-linear elements of matrix $[C]^{t+\Delta t}$, and resolve for mere accurate values of $h_J^{t+\Delta t/2}$. The solving process is repeated until the change in successive values of $h_J^{t+\Delta t/2}$ is within a prescribed head tolerance. When convergence is ensured, the last set of values of $h_J^{t+\Delta t/2}$ may be substituted into equation (4-79) to obtain the nodal values at $t+\Delta t$.

4.2.5 Treatment of Conditions on the Well Boundary

In solving the problem of flow toward a pumped well, special treatment must be given to the conditions prevailing on the well boundary. Two types of well boundary condition are possible, depending on the pumping operation. If the well is pumped at a constant discharge, the condition of prescribed flow rate will prevail. On the other hand, if it is pumped such that the pumping water level is held constant, the prescribed head condition will result. These two types of condition may be dealt with in the following manner.

(i) Prescribed Head Condition

Consider a typical pumped well shown in Fig.(4-b). As indicated in the figure, the first portion of the well boundary is screened and the remaining portion is cased. If the water level is maintained constant throughout the pumping period, the head values at the nodes situated on the well screen will be constant with time and equal to the known elevation of the water level above the datum plane.

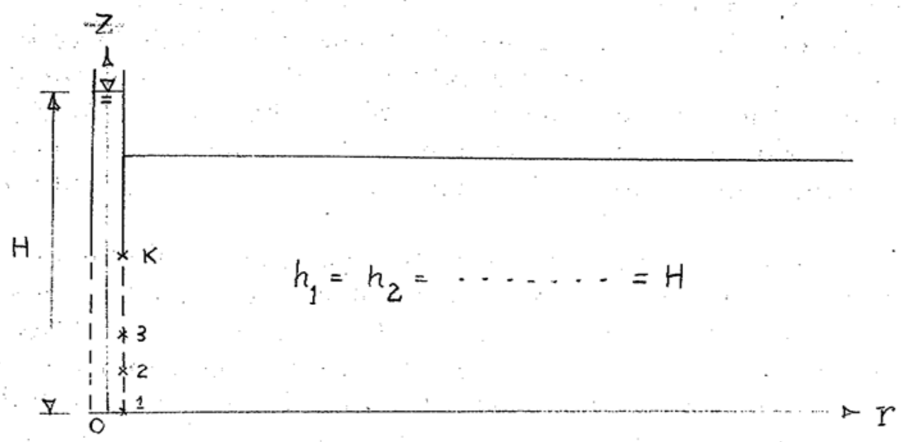

Fig. (4-6): The boundary of a typical pumped well.

In order to incorporate the prescribed head condition into equation

(4-78), the following scheme for partitioning the gross matrices in this equation is employed.

Preserving the capital and small letter subscript notation, let α and j be a Greek letter and a small letter subscripts referring to the nodes situated on the well screen and the remaining nodes in the flow region respectively. Hence it follows that α ranges from 1 to K, j ranges from K+1 to n, and equation (4-79) may be expanded. On introducing $X_J = h_J^{t+\Delta t/2}$ and expanding subscript J, the following equation results.

$$\left(\frac{\Delta t}{2} C_{\alpha I} + D_{\alpha I}\right) X_\alpha + \left(\frac{t}{2} C_{jI} + D_{jI}\right) X_j = D_I h + D_{jI} h_j - \frac{t}{2} F_I \qquad (4\text{-}80)$$

where the superscripts have been dropped from matrices

$$C^{t+\Delta t/2} \quad , \quad F^{t+\Delta t/2} \quad \text{and} \quad h^t \quad .$$

Now the prescribed head conditions require that

$$x_\alpha = h_\alpha = H \qquad (4\text{-}81)$$

for = 1 ....K

Substituting the above equation into equation (4-80) and rearranging gives

$$\left(\frac{\Delta t}{2} C_{jI} + D_{jI}\right) X_j = D_{jI} h_j - \frac{\Delta t}{2} C_{\alpha I} h_\alpha - \frac{\Delta t}{2} F_I \qquad (4\text{-}82)$$

By expanding subscript I, equation (4-82) can be expanded to give the following equations:

$$\left(\frac{\Delta t}{2} C_{ji} + D_{ji}\right) X_j = D_{ji} h_j - \frac{\Delta t}{2} C_{\alpha i} h_\alpha - \frac{\Delta t}{2} F_i \qquad (4\text{-}83)$$

and

$$\left(\frac{\Delta t}{2} C_{j\beta} + D_{j\beta}\right) X_j = D_{j\beta} h_j - \frac{\Delta t}{2} C_{j\beta} h_j - \frac{\Delta t}{2} F_\beta \qquad (4\text{-}84)$$

where j is a small letter subscript having the same range as i, and β is a Greek letter subscript having the same range as α .

Equation (4-83) represents a system of n-K equations. If the non-linear coefficients, C_{ji}, are evaluated in terms of the known nodal head values, the equations may be solved for X_j by employing the gaussian elimination process. The non-linear coefficients, C_{ji}, can then be re-calculated in terms of the current values of X_j, and the equations resolved until convergence results.

Also if required, the values of the flux at the nodal points along the well screen can be calculated by substituting the last set of values of X_j into equation (4-84) and solving for F

Thus from equation (4-84)

$$F_\beta = \frac{2}{\Delta t} D_{j\beta} h_j - C_{j\beta} h_j - (C_{j\beta} + \frac{2}{\Delta t} D_{j\beta}) X_j \qquad (4\text{-}85)$$

The total well discharge Q is given by

$$Q = \left(\sum_{\beta=1}^{K} F_\beta \right) \qquad (4\text{-}86)$$

(ii) <u>Prescribed Flow Rate Condition</u>

In the extraction of groundwater by pumping, it is common practice to maintain constant total discharge from the well throughout the pumping period. Accordingly, since the total flow rate is fixed, the water level in the well and hence the hydraulic head along the pervious portion of the well boundary must vary with time.

Consider the well shown in Fig. (4-6). If $\bar{Q}$ is the prescribed flow rate, the prescribed flow rate condition will be given by

$$\bar{Q} = \sum_{\beta=1}^{K} F_\beta \qquad (4\text{-}87)$$

where F_α are nodal flux values.

Also the requirement of constant head distribution along the well screen at time $t + \Delta t/2$ is given by

$$h_1^{t+\Delta t/2} = h_1^{t+\Delta t/2} = \ldots\ldots\ldots = H^{t+\Delta t/2} \qquad (4\text{-}88)$$

where $H^{t+\Delta t/2}$ is the unknown height of the water level at time $t+\Delta t/2$.

In the general case where the distribution of flux along the well screen is non-uniform, the prescribed flow rate condition is satisfied by $t+\Delta t/2$ adjusting the value of $H^{t+\Delta t/2}$ until equation (4-87) is approximately satisfied. The procedure is to assume a value of $H^{t+\Delta t/2}$, proceed in the same way as in the case of the prescribed head condition and solve for F_β , then check if equation (4-87) is satisfied within the prescribed tolerance of $\bar{Q}$. If the equation is not satisfied the value of $H^{t+ \quad t/2}$ is adjusted by assuming that the total well discharging rate is proportion to the well drawdown, which is defined as the difference between the initial height of the water table and the height of the pumping water level.

In the simpler case of flow where the distribution of flux along the well screen is known to be uniform, the prescribed flow rate condition may be incorporated into equation (4-78). The detailed treatment has been presented by Javandel and Witherspoon (1968).

4.2.6 Elimination Scheme for Solving a System of Linear Equations

The assemblage of element matrices leads to a system of n simultaneous equations which, after imposing the conditions prevailing on the well boundary, reduces to a system of n-K equations as represented by equation (4-83). The reduced system is linearised by evaluating the non-linear coefficients C_{ji} in the equations in terms of the known nodal values of the hydraulic head.

A banded elimination scheme is employed to solve for the n-K unknowns in the linear system of equations. The scheme takes into account the sparseness and symmetry of the gross matrices, [C] and [D]. The two matrices are arranged in compact banded form by numbering the nodes in the flow region in consecutive order. Proper numbering reduces the bandwidth to a minimum. The process of elimination is accomplished by reducing the system of equations to an equivalent triangular form through a series of arithmetic operations on the coefficients of the equations. Then, starting from the last equation the last unknown is solved, and the remaining unknowns are obtained by the process of back substitution into the previous equations.

Due to symmetry of matrices [C] and [D], it is only necessary to operate on the elements in their upper triangles. The half band-width of each row is computed as the length between the diagonal element and the last non-zero element in the row. In a computer subroutine developed, the two matrices are converted into gross vectors by stringing together the half-bands of all successive rows. This conversion partly eliminates the problem of insufficient computer capacity, as only a small part of the two gross matrices need to be stored. Furthermore, a smaller number of necessary arithmetic operations considerably cuts down the solution time.

5. Conclusions and Further Work

Hydraulic principles and field equations applicable to transient, three dimensional, two regime well flow have been developed. The two flow regimes, namely Darcy and non-Darcy flow regimes, were considered to be distinct. Transition from Darcy to non-Darcy flow was indicated by the critical Reynolds number, and non-Darcy flow was described by the Forchheimer non-linear velocity-gradient equation of flow through isotropic porous media.

A generalised variational principle for transient, two regime well flow through confined and unconfined aquifers has been established. Energy approach to well flow problems has been proposed, and an energy theorem has been stated, proved and directly related to the variational principle.

A general formulation of the finite element method for analysing three-dimensional confined flow has been presented. Axi-symmetric confined flow problems have been solved by using triangular and isoparametric finite elements. Typical results obtained for steady state flow will be presented in Section C.

It is recommended that additional research should be carried out to attain the following aims:-

(i) To develop a better understanding of flow transition and a more satisfactory criterion to describe the onset of non-Darcy flow than the critical Reynolds number used in the present study.

(ii) To investigate non-Darcy flow through anisotropic porous media and develop macroscopic, non-linear, velocity-gradient relations describing the flow.

(iii) To extend the generalised variational principle and energy theorem to describe two regime well flow through anisotropic aquifers.

(iv) To extend the finite element formulation of confined flow problems presented to unconfined flow problems.

List of References.

1. Boulton, N.S. - "Analysis of data from non equilibrium pumping tests allowing for delayed yield from storage", Proc. Inst. Civil Engrs. (U.K.), Vol. 26, No. 6693, (1963).

2. De Wiest, R.J.M. - "Flow through porous media", Academic Press, New York, (1969).

3. Dudgeon, C.R. - "Flow of water through coarse granular materials", M. Eng. thesis, Univ. of New South Wales, (1964).

4. Engelund, F. - "On the laminar and turbulent flows of groundwater through homogeneous sand", Hydraulic Lab., Tech. Univ. of Denmark, Bull. No. 4, (1953).

5. Ergatoudis, O.C. et al - "Curved isoparametric, quadrilateral elements for finite element analysis", Int. J. Solids Struct., 4, 31-42, (1968).

6. Fenton, J.D. - "Hydraulic and stability analysis of rockfill dams", Dept. of Civil Eng., Univ. of Melbourne, Australia (1968).

7. Hantush, M.S. - "Hydraulics of wells", Advances in Hydroscience, edited by Ven Te Chow, Academic Press, New York, Vol. 1, (1964).

8. Huyakorn, P.S. and Dudgeon, C.R. - "Groundwater and well hydraulics:- An annotated bibliography", Water Research Lab., Univ. of New South Wales, Report No. 121, Feb., (1972).

9. Jacob, C.E. - "Flow of groundwater", Engineering Hydraulics edited by Rouse, H., Wiley, New York, (1950).

10. Javandel, I. and Witherspoon, P.A. - "Analysis of transient fluid flow in multi-layered systems", Dept. of Civil Engineering, Univ. of California, Berkeley, March, (1968).

11. McCorquodale, J.A. and Ng, H.C. - "Non-Darcy flow solved by finite element analysis", I.A.H.R., 13th Congress, Proc., Vol. 4, Subj. D, pp. 347-355, Kyoto, (1969).

12. McCorquodale, J.A. - "Variational approach to non-Darcy flow", A.S.C.E., Jour., 96, HY11, pp. 2265-78, Nov., (1970).

13. Mauersberger, P. - "A variational principle for steady state groundwater flow with a free surface", Pure Appl. Geophys., 60, 101, (1965).

14. Mogg, J.L. - "Effect of aquifer turbulence on well drawdown," A.S.C.E, Jour., 85, HY11, Nov., (1959).

15. Muskat, M. - "The flow of homogeneous fluids through porous media", McGraw-Hill, New York, (1937).

16. Neuman, S.P. and Witherspoon, P.A. - "Transient flow of groundwater to wells in multiple-aquifer systems", Dept. of Civil Eng., Univ. of California, Geotech. Eng., Publ. No. 69-1, (1969).

17. Neuman, S.P. and Witherspoon, P.A. - "Variational principles for confined and unconfined flow of groundwater, Water Resour. Res., 6, 5, pp. 1377-82, 1970.

18. Neuman, S.P. and Witherspoon, P.A. - "Variational principles for fluid flow in porous media", A.S.C.E., Jour., 97, E.M.2 pp. 359-374, April (1971).

19. Parekh, C.J. - "A finite element solution of time dependent field problems", M.Sc. thesis, Univ. of Wales, Swansea, (1967).

20. Rorabaugh, M.I. - "Graphical and theoretical analysis of step-drawdown test of artesian wells", A.S.C.E., Trans., 79, pp. 362, (1953).

21. Stark, K.P. and Volker, R.E. - "Non-linear flow through porous materials", Dept. of Eng., Univ. College of Townsville, Australia, Res. Bull. No. 1, April (1967).

22. Sunada, D.K. - "Turbulent flow through porous media", Hydraulic Lab., Univ. of California, Berkeley, Contribution No. 103, (1965).

23. Taylor, R.L., Brown, C.B. - "Darcy flow solutions with a free surface" A.S.C.E., Jour., 93, HY2, 25-33, March (1967).

24. Trollope, D.H. et al - "Complex flow through porous media", Dept. of Eng., James Cook, Univ. of North Queensland, Australia, May (1970).

25. Verruijt, A. - "Theory of groundwater flow", McMillan and Co., London, (1970).

26. Volker, R.E. - "Nonlinear flow in porous media by finite elements", A.S.C.E., Jour., 95, HY6, 2093-2112, (1969).

27. Walton, W.C. - "Groundwater resource evaluation", McGraw-Hill Book Co., New York, (1970).

28. Wright, D.E. - "Non-linear flow through granular media", A.S.C.E., Jour., 94, HY4, pp.851-872, July 1968.

29. Zienkiewicz, O.C. and Cheung, Y.K. - "Finite element in the solution of field problems", The Engineer (U.K.) pp. 220, Sept., (1965).

30. Zienkiewicz, O.C. and Cheung, Y.K. "The finite element method in structural and continuum mechanics", McGraw Hill Book Co., London, (1967).